

ANALYSIS OF THE REACTION $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$ AT 25 AND 40 GeV/c

Inaugural-Dissertation

zur Erlangung der Doktorwürde

der Fakultät für Physik
der Ludwig-Maximilians-Universität
München

Robert Klanner
Genf, 1973

ANALYSIS OF THE REACTION $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ AT 25 AND 40 GeV/c

Inaugural-Dissertation

zur Erlangung der Doktorwürde

der Fakultät für Physik
der Ludwig-Maximilians-Universität
München

Robert Klanner

Genf, 1973



1. *Berichterstatter:* Prof. Dr. U. Meyer-Berkhout
2. *Berichterstatter:* Prof. Dr. N. Schmitz

Tag der mündlichen Prüfung: 13.4.1973

ABSTRACT

This work describes a counter experiment, which measured the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ at beam momenta of 25 and 40 GeV/c. The experiment was performed at the 76 GeV/c proton synchrotron at Serpukhov, USSR.

The events of the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ have been measured by spark chambers, scintillation counters and on analysing magnet, and identified by kinematic reconstruction. Production and decay of the 3π system has been investigated by the partial wave decomposition of the 3π system. The following results have been found:

- i) The 3π systems A1 and A2, known from experiments at lower beam energies, are strongly produced at high energies.
- ii) Whereas the A2 system can be described by a Breit-Wigner amplitude, this is not true for the A1 system.

The results of this experiment are compared with the findings of other experiments.



Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41555409_0048

CONTENTS

	Page
1. INTRODUCTION	1
2. APPARATUS, CALIBRATIONS, AND ACCEPTANCE	7
2.1 THE TRIGGER	7
2.2 BEAM AND TARGET	8
2.3 PROTON TELESCOPE	9
2.4 MAGNETIC SPECTROMETER	11
2.5 3π EFFECTIVE MASS RESOLUTION AND ACCEPTANCE OF THE APPARATUS	14
2.6 DATA ACQUISITION, ON-LINE COMPUTER, AND DATA ANALYSIS	14
3. SELECTION OF EVENTS OF THE CHANNEL $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$	31
3.1 SELECTION OF EVENTS $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$	31
3.2 CONTAMINATIONS FROM OTHER CHANNELS	33
3.3 CROSS-SECTION NORMALIZATION	34
4. DISCUSSION OF THE MAIN FEATURES OF THE DATA	47
4.1 EFFECTIVE MASS PLOTS	47
4.2 CROSS-SECTION FOR THE REACTION $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ VERSUS 3π EFFECTIVE MASS, INCIDENT BEAM MOMENTUM, AND MOMENTUM TRANSFER	48
5. METHOD OF PARTIAL-WAVE ANALYSIS	59
5.1 QUANTUM MECHANICAL DESCRIPTION OF THE REACTION $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$	59
5.2 FIT OF THE THEORETICAL DISTRIBUTION TO THE DATA	67
6. RESULTS OF THE PARTIAL-WAVE ANALYSIS BELOW A 3π MASS OF 1.5 GeV	73
6.1 PARTIAL-WAVE ANALYSIS OF THE DATA	73
6.2 GENERAL RESULTS OF THE PARTIAL-WAVE ANALYSIS	80
6.3 RESULTS CONCERNING THE PRODUCTION AND DECAY OF THE A_1 MESON	82

	Page
6.4 RESULTS CONCERNING THE PRODUCTION AND DECAY OF THE A_2 MESON	86
6.5 DISCUSSION OF THE RESULTS	88
Acknowledgements	115

(The references, tables and figures appear at the
end of each chapter)

CHAPTER 1

INTRODUCTION

There are at present two complementary approaches to the experimental and theoretical investigation of reactions at high energies (arbitrarily defined as centre-of-mass energies around or above 5 GeV).

In the inclusive approach one- or two-particle distributions are studied, disregarding all other particles or particle systems produced in the interaction.

In the exclusive approach, all particles produced in the interaction are measured and particular reaction channels are identified, with the aim of describing the collisions in all kinematically relevant variables. The exclusive analysis, which is relatively simple for reactions where two particles are produced, becomes however more and more complicated with increasing multiplicity n , as the number of dimensions of phase space grows like $3n - 3$. Two- and three-body reactions were extensively studied by bubble-chamber and counter experiments, whereas the investigation of the more complex four- and many-body reactions has been almost entirely confined to bubble-chamber experiments. Until now the exclusive analysis of few-body reactions ($n > 2$) has been performed at centre-of-mass energies up to 7 GeV, corresponding to a beam particle of 25 GeV/c hitting a stationary proton target.

Here we present results from a counter experiment performed at the 76 GeV Proton Accelerator at Serpukhov USSR by a collaboration of the European Organization for Nuclear Research (CERN), the Russian Institute for High-Energy Physics at Serpukhov (IHEP), and the Universities of Geneva, Illinois at Urbana-Champaign, and Munich. The aim of the experiment was to search for heavy boson resonances with masses up to 4 GeV. We investigated the inclusive reaction

$$\pi^- p \rightarrow X^- p$$

(X^- stands for any unknown object) with the missing-mass technique^{*)} at 25 and 40 GeV/c incident pion momentum. The results have been published elsewhere^{1,1)}. In Fig. 1.1 we show the measured differential cross-section $d^2\sigma/dtdM_{X^-}^2$ as a function of $M_{X^-}^2$, the mass of X^- squared at a momentum transfer to the proton t of 0.25 (GeV/c)^2 . At low masses below 2 GeV the cross-section varies rapidly with mass, suggesting the excitation of meson resonances. At high masses above 2 GeV it becomes a smooth function of the missing mass M_{X^-} , and we see no evidence for resonances.

In this work we analyse the particular decay mode

$$X^- \rightarrow \pi^- \pi^- \pi^+$$

(i.e. the exclusive reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$) for masses of X^- below 1.5 GeV, to identify the structures of the missing-mass spectrum. We are interested in the following questions:

- Are they resonances; what are their quantum numbers?
- How are they produced?

We study the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ by performing a partial-wave analysis of the 3π system. Similar to the partial wave analysis of low-energy production experiments like $\pi p \rightarrow N^* \rightarrow \pi p$ we determine the intensities and polarizations of the different partial waves and the relative phases between them. Thus we find the quantum numbers of these structures and can test whether they are resonances or not.

To find out the production mechanism of the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ we study the energy dependence of the differential cross-section as well as the quantum numbers of the produced 3π system. Usually high-energy reactions of the type $a + b \rightarrow a' + b'$, where a and b are beam and target particle, a' and b' are single hadrons or hadron systems, are classified according to the quantum numbers exchanged in the interaction.

*) In the missing-mass technique only the beam particle and the recoil proton are measured, and the mass of the missing object X^- is calculated using energy conservation. The information about X^- is thus limited to its mass and the quantum numbers conserved in the interaction.

If the internal quantum numbers ^{*)} of a and a' and of b and b' are the same, i.e. no internal quantum numbers are exchanged, the reaction cross-sections are nearly energy-independent. Due to the similarity of these reactions to elastic scattering ($a = a'$ and $b = b'$), they are frequently called quasi-elastic or diffractive processes.

If internal quantum numbers are exchanged the internal quantum numbers of a and a' , or of b and b' , are different, and the reaction cross-sections decrease as one over the beam momentum or faster.

We first describe the apparatus in Chapter 2 and the selection of events in Chapter 3. After reviewing briefly the main features of our data sample in Chapter 4, we will develop the formalism which we use to perform the partial wave decomposition of the 3π system in Chapter 5. In Chapter 6 we finally discuss the results of the analysis.

*) The internal quantum numbers are baryon number, electric charge, strangeness, isospin, and G-parity, the last quantity being defined for non-strange mesons only.

REFERENCES

- 1.1) Y.M. Antipov, R. Baud, R. Busnello, G. Damgaard,
M.N. Kienzle-Focacci, W. Kienzle, R. Klanner, L.G. Landsberg,
A.A. Lebedev, C. Lechanoine, P. Lecomte, M. Martin,
V. Roinishvili, R.D. Sard, A. Weitsch and F.A. Yotch, Phys.
Letters 40 B, 147 (1972).

FIGURE CAPTION

Fig. 1.1 : Missing-mass X^- spectrum measured at 25 and 40 GeV/c in
the reaction $\pi^- p \rightarrow X^- p$ in the momentum transfer interval
between 0.17 and 0.35 (GeV/c)². The data are from
Reference 1.1.



Fig. 1.1

CHAPTER 2

APPARATUS, CALIBRATIONS, AND ACCEPTANCE

In this chapter, the experimental set-up of the CERN-IHEP Boson Spectrometer^{2.1)} used to measure the reaction

$$\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$$

and how the different parts of the experiment were calibrated and monitored are described. Figures 2.1 and 2.2 show the layout of the apparatus and explain the symbols used in the text for the different parts of the experiment. Table 2.1 summarizes the essential results of this chapter.

2.1 THE TRIGGER

The trigger selects events with an incoming beam particle interacting inside the liquid hydrogen target, and a slow recoil proton leaving the target. The scintillation counters T1 and T2, and the anticounter $\overline{T0}$ define the incoming particle pointing to the target. The anticounters $\overline{B1}$ and $\overline{B2}$, downstream of the apparatus, record that the beam particle has interacted in the target. The proportional wire chamber PRCH close to the target and the counter R3 detect the recoil particle, the anticounters $\overline{P14} \cdot \overline{P15}$ ensure that it has stopped in one of the range counters P0-P13. By demanding that the recoil particle arrives at the R3 counter about 4 nsec later than particles with the velocity of light "slow" recoil particles (i.e. velocity $\beta < 0.7c$) are selected. The full trigger condition is thus:

$$T1 \cdot T2 \cdot \overline{T0} \cdot \overline{B1} \overline{B2} (R3 \cdot \overline{P14} \overline{P15}) PRCH$$

Depending on the amount of material in front of the trigger counter R3 of the proton telescope, three different conditions, classified according to the momentum transfer t to the proton, have been chosen.

Condition	Momentum interval [GeV/c]	t interval [(GeV/c) ²]
Low t	0.200-0.600	0.04-0.33
Medium t	0.320-0.600	0.10-0.33
High t	0.420-0.600	0.17-0.33

In some runs two additional anticounters E1 and E2 were used in anticoincidence to suppress elastic events.

Depending on beam intensity and background conditions, 70-90% of the triggers were good events with an incoming beam particle in the hodoscopes and a recoil particle in the spark chambers CH5 and CH6. The dead-time losses of the B1B2 anticoincidence were below 1% up to 10^6 incident particles per second. The efficiency of the proportional wire chamber was 97-98% and practically independent of proton momentum.

2.2 BEAM AND TARGET

The experiment used the 25-40 GeV/c unseparated beam 4B, derived at zero degree production angle from an internal target of the 76 GeV/c Proton Accelerator at Serpukhov (USSR). The following incident particles rates were counted by the beam counters, and identified by the four threshold Čerenkov counters.

Momentum [GeV/c]	Intensity (at $\pm 1\%$ momentum spread)	Particles ratios			
		μ^-	π^-	K^-	$\bar{p}$
25	10^6	~ 0.01	0.96	0.021	0.009
40	3×10^5	~ 0.01	0.97	0.018	0.002

The two scintillation counter hodoscopes H1 and H2 measure the track of the incident particle to ± 2 mm and ± 0.5 mrad ^{*)}. The momentum of the beam particles is determined from their position in the counter hodoscopes H0, H1, and H2 and the magnetic beam elements between H0 and H1. In a test run the mean momentum for every hodoscope element combination is measured by the magnetic spectrometer described in Section 2.4

*) One standard deviation.

and an additional chamber (CH7) 12 m behind the magnet. From the size of the hodoscope elements and the multiple scattering of the particles in the counters we calculate $\pm 0.3\%$ momentum resolution. The absolute momentum is known to better than $\pm 1.2\%$.

The beam is focused (± 3 mm horizontally and ± 5 mm vertically) onto the liquid hydrogen target (43 cm long, 4 cm diameter) 6-10 mm away from its wall towards the proton telescope, as described in the next section.

2.3 PROTON TELESCOPE

With the proton telescope we identify slow recoil protons produced by the incident particles in the liquid hydrogen, and measure their momenta p and their scattering angles θ . The telescope consists of a proportional wire chamber PRCH, two magnetostrictive wire chambers CH5-CH6, a time of flight and trigger counter R3, and a range-counter telescope P0-P15 (Fig. 2.3), mounted on a turn-table, which can be moved into the beam for calibration purposes.

2.3.1 Measurement of the proton momentum

The proton is identified and momentum analysed by the combined measurement of its velocity and its range.

The 150×80 cm² large scintillation counter R3 is seen by two photomultipliers from above and from below. The time of flight (TOF) of the beam particle between the T2 counter (125 cm in front of the target) and the interaction point, plus the time of flight of the recoil particle to the R3 counter (280 cm away from the target), is thus measured by two independent circuits. From the difference between both measurements we find ± 0.3 nsec time resolution. The following corrections to the measured time of flight are applied to obtain the velocity of the proton at the interaction point in the target:

- i) The time of flight of the incident particle between the counter T2 and the interaction vertex is subtracted.
- ii) The time delay due to the propagation of the light signal in the R3 counter is subtracted. This correction has been measured using relativistic pions.

iii) The time of flight is corrected for the slowing down of the protons in the liquid hydrogen. This correction, which can be as large as 30 MeV/c for 200 MeV/c protons, is calculated according to the tables of C. Serre^{2.2}).

The calibration and resolution of the velocity measurement is obtained from $\pi^- p \rightarrow \pi^- p$ elastic events. The error on the momentum determination by the time-of-flight measurement can be calculated from the ± 0.3 nsec measurement error and the ± 2 mm uncertainty of the proton path in the liquid hydrogen, or measured by the following method.

From a fully measured elastic event the recoil momentum of the proton can be determined in several ways:

- from the TOF measurement: p_m ,
- from the beam momentum and the proton recoil angle: p_a ,
- from the beam momentum and the scattering angle of the forward particle: p_b .

With the measured standard deviations of the following distributions:

$$\begin{aligned}\sigma_a^2 &= \sigma^2(p_m - p_a) = \sigma^2(p_m) + \sigma^2(p_a) \\ \sigma_b^2 &= \sigma^2(p_m - p_b) = \sigma^2(p_m) + \sigma^2(p_b) \\ \sigma_{ab}^2 &= \sigma^2(p_a - p_b) = \sigma^2(p_a) + \sigma^2(p_b)\end{aligned}$$

we obtain for the uncertainty on the measured recoil momentum:

$$\sigma^2(p_m) = \frac{1}{2} (\sigma_a^2 + \sigma_b^2 - \sigma_{ab}^2) .$$

In Fig. 2.4 we compare $\sigma(p_m)$ with the calculated error on the proton momentum determination. The calibration of the range telescope, as well as the uncertainty of the momentum measurement by range, is done in the same way.

Some of the protons undergo nuclear interactions in the range counters and lose so much energy that range and velocity measurement do not agree. This will be discussed in more detail in Chapter 3

(Section 3.2). Here we note only that we use for the further analysis the weighted average of both measurements if they agree within two standard deviations, and that we use time of flight if they do not.

2.3.2 Measurement of the proton scattering angle

The direction of the recoil proton is measured by two magnetostrictive wire spark chambers (six planes). The geometrical resolution of the chambers is small compared to the contribution of multiple scattering to the measurement error. With the method described in Section 2.3.1 we determine the angular resolution. The result is shown in Fig. 2.5 and compared to the multiple scattering calculated according to the formula

$$\sigma_{\theta} = C / p \beta$$

where $C = 0.9 \pm 0.1$ mrad/(GeV/c) is obtained from the amount of material in the proton path.

From $\pi^- p$ elastic events, identified by an incoming pion and a forward scattered particle, we determine the efficiency of the spark chambers and the track finding program to be $99.3 \pm 0.3\%$.

2.4 MAGNETIC SPECTROMETER

The magnetic spectrometer is used to measure the three pions in the reaction

$$\pi^- p \rightarrow \pi^- \pi^- \pi^+ p .$$

It consists of three wire chambers in front of and behind an analysing magnet.

2.4.1 Wire chambers, resolution, and efficiency

The construction and performance of the magnetostrictive wire chambers (Fig. 2.6) is described in detail elsewhere^{2,3}). The sensitive surface of the chambers is 150×150 cm². In the centre of the chambers there is a dead spot to avoid recording beam tracks. Each chamber consists of four wire planes to measure four different projections (x, y, and u and v turned by 10°) of a particle track. The three chambers of one

pair are aligned such that x, y, u and v planes are parallel. The track-reconstruction program first searches for tracks (i.e. three sparks on a straight line) in the four projections and then combines the different projections in space.

In a test run without magnetic field the alignment of all wire chambers is calibrated to ± 0.1 mm with beam tracks measured in the hodoscopes H1 and H2. From the measured deflection in the absence of a magnetic field we deduce the angular resolution $\sigma(\theta)$ of the spectrometer. It depends on the inclination θ of the tracks to the normal of the chamber

$$\sigma_{\theta} [\text{mrad}] = 0.5 + 30 \vartheta^2 [\text{rad}^2] .$$

The over-all (i.e. the product of chamber and track-finding) efficiency for both chamber pairs is continuously measured by the $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ events. To obtain the efficiency of the chambers in front of the magnet we select tracks that are measured in the chambers behind the magnet and point in the vertical direction, where the deflection is small, to the vertex in the liquid hydrogen target. From the vertex position and the position and direction behind the magnet we predict its position in the chambers in front of the magnet. Counting how often the track has been found yields the efficiency ϵ . The efficiency ϵ depends to a good approximation on the inclination θ of the tracks only.

To measure the efficiency of the chambers behind the magnet we select events with a recoil proton and three particles in the forward system, of which at least one is momentum analysed. Using the three constraints of momentum conservation we can check if we really have a four-particle event. If we do, we calculate the missing momenta and predict the position of the tracks in the chambers behind the magnet. Counting how often these chambers failed to measure the tracks which passed through the gap of the magnet yields the inefficiency $1 - \epsilon$ of this chamber group. Figure 2.7 shows the efficiencies for both pairs of chambers for the 40 GeV/c October 1971 run.

2.4.2 Magnet and momentum measurement

The magnet has a gap of 50 cm height and 155 cm width. Operated at a field of 13 kG its bending power is 18.5 kGm. The three components of the magnetic field have been measured at 80,000 points by Hall plates

with a precision on the main component of $\pm 0.2\%$. In order to obtain the momentum p of a particle from its measured quantities x_i , where $i = 1, \dots, 5$ (we use its position and direction in a plane in the centre of target and the horizontal deflection), we integrated 25,000 tracks through the magnet to obtain an array:

$$p^j, x_1^j, x_2^j, x_3^j, x_4^j, x_5^j \quad j = 1 \dots 25\,000.$$

This table is used to determine by a least squares fit the expansion coefficients of the expression

$$\frac{1}{p} = \sum_{ijklm} C_{ijklm} \cdot T_i(x_1) \cdot T_j(x_2) \cdot T_k(x_3) \cdot T_l(x_4) \cdot T_m(x_5)$$

For the polynomials T we used Chebyshev polynomials as did Lechanoine et al.^{2.4)}. For our magnet and our geometry 144 coefficients are sufficient to ensure that for all tracks the difference between the momentum calculated from the expansion and from integration through the field is less than 0.3%. A sample of 2500 $K_S^0 \rightarrow \pi^+ \pi^-$ decays measured at 25 GeV/c has been used to check the momentum reconstruction. Figure 2.8 shows the $\pi^+ \pi^-$ effective mass spectrum for events with a $\pi^+ \pi^-$ vertex more than 3.5 cm downstream of the proton vertex. Comparing the fitted mass value of

$$m_{K^0} = (498.1 \pm 0.2) \text{ MeV}$$

with the world average^{2.5)} of 497.8 ± 0.15 , we deduce that we know the field integral to better than $\pm 0.2\%$ (an over-all change of the magnetic field of 1%, shifts the K mass by 3.2 MeV). The measured width of the K^0 peak

$$\Gamma_{\text{exp}} = (11.2 \pm 0.4) \text{ MeV}$$

agrees with the width of 10.8 MeV, calculated from the angular resolution of the spark chambers. No systematic distortions could be found when selecting events with tracks passing through different regions of the magnet.

2.5 3 π EFFECTIVE MASS RESOLUTION AND ACCEPTANCE OF THE APPARATUS

In the last sub-section we discussed how the resolutions of all measured quantities were determined. To calculate the 3 π effective mass resolution for the $\pi p \rightarrow \pi^- \pi^- \pi^+ p$ events, we perform a kinematic fit with the method described by Ronne^{2,6)} to obtain the best estimates for the momentum vectors of all particles and their error matrix. From the error matrix we can evaluate the error on all quantities calculated from the momenta. Figure 2.9 shows the 3 π effective mass resolution for events with three or two of the pions momentum analysed.

To demonstrate the acceptance of our system, Fig. 2.10 shows the boundaries of our apparatus in the $\pi^- p$ centre-of-mass frame for 40 GeV/c incident momentum. On the left side a horizontal and on the right side a vertical cut through the apparatus is displayed. The dashed surface to the left is the acceptance of the recoil spectrometer. We see that we accept protons which are produced backwards in the centre-of-mass system with a transverse momentum of ~ 300 MeV/c. The rightmost sector is the acceptance of the magnet and the chambers behind the magnet for negative pions. Whenever a pion is produced in this part of phase space it will be momentum-analysed in the spectrometer. If it is produced in the region between the dashed lines, which represent the aperture of the chambers in front of the magnet and the magnet aperture, only its direction will be measured. From this figure we see that the experiment is well suited to study the forward production of low mass 3 π systems.

2.6 DATA ACQUISITION, ON-LINE COMPUTER, AND DATA ANALYSIS

Figure 2.11 shows a schematic diagram of the data flow from the apparatus to the final results. The information from the experiment (i.e. the propagation time of the magnetostrictive signals in the read-out lines, the signals from the scintillation counters, etc.) is digitized into 425 16-bit scalars. The scalars are read by an IBM 1800 computer, which writes the scaler content onto magnetic tape, and performs a detailed technical monitoring of the apparatus (efficiency of the spark chambers, counting rates of the scintillation counters, position of the beam in the beam hodoscopes, etc.). For a detailed description

of the on-line computer see the thesis of R. Baud^{2.7)}. The tapes are sent by air to Geneva, where the further analysis is performed on the CDC computers of CERN and of the University of Geneva. In the "Raw-Analysis Program", the information from the scalers is decoded, and position, direction and momenta of the measured particles are reconstructed. Next, the events go through the "Event-Reconstruction Program" (see Chapter 3), which selects events of the channel $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ and identifies them by a kinematic fit. We finally extract the physical results from the data by the "Partial-Wave Method", described in Chapter 5.

REFERENCES

- 2.1) A. Weitsch, The CERN-IHEP Boson Spectrometer *in* Proc. 2nd Philadelphia Conference on Meson Spectroscopy 1970 (Ed. H. Baltay), p. 557.
- 2.2) C. Serre, Evaluation de la perte d'énergie unitaire et du parcours de particules chargées traversant un absorbant quelconque, CERN 67-5 (1967).
- 2.3) G.E. Chikovani, M. Fischer, M.N. Focacci, W. Kienzle, G. Laverrière, C. Lechanoine, B. Levrat, M. Martin and P. Schübelin, Test results of digitized wide-gap wire chambers, CERN 67-13 (1967).
G.E. Chikovani, G. Laverrière and P. Schübelin, Nuclear Instrum. Methods 47, 273 (1967).
- 2.4) C. Lechanoine, M. Martin and H. Wind, Nuclear Instrum. Methods 69, 122 (1969).
- 2.5) N. Barash-Schmidt, A. Barbaro-Galtieri, J. Bartels, C. Bricman, V. Chaloupka, J. Enstrom, T. Lasinsky, A. Rittenberg, M. Roos, P. Söding, T. Trippe, Rev. Mod. Phys. 43, No. 2, Part II (1971).
- 2.6) B. Ronne, Kinematical analysis of bubble chamber pictures *in* Proc. Herceg-Noví Summer School, 1964, CERN 64-13 (1964), p. 87.
- 2.7) R. Baud, Acquisition de données en physique corpusculaire, Thèse 1575, Université de Genève, 1971.

TABLE CAPTION

Table 2.1 : Summary of resolution and efficiency of the different parts of the CERN-IHEP Boson Spectrometer.

FIGURE CAPTIONS

Fig. 2.1 : Schematic layout of the CERN-IHEP Boson Spectrometer.

Explanation of symbols used:

H : beam hodoscope,

Č : beam Čerenkov counter,

T : beam counter,

PRCH: proportional wire chamber,

CH : spark chamber,

E : elastic counter,

B : anti-beam counter,
 R : time-of-flight trigger counter,
 P : range counter,
 : focusing, defocusing quadrupole,
 : bending magnet,
 : momentum slit.

Fig. 2.2 : Photograph of the CERN-IHEP Boson Spectrometer.

Fig. 2.3 : Photograph of the range telescope.

Fig. 2.4 : Proton momentum resolution. The dots are the resolution from the time-of-flight measurement, the solid curve the calculated value. The crosses are the resolution of the momentum determination by range.

Fig. 2.5 : Proton angular resolution. The measurement error is dominated by multiple scattering of the protons in the liquid hydrogen, target walls, and chamber. It amounts to 3.7 mrad at 500 MeV/c momentum.

Fig. 2.6 : Photograph of the wire spark chambers during a burst.

Fig. 2.7 : a) Efficiency of the spark chambers in front of the magnet
 b) Efficiency of the spark chambers behind the magnet.

Fig. 2.8 : Test of the magnetic spectrometer by measuring the $\pi^+\pi^-$ decay of K^0 mesons in the reaction $\pi^-p \rightarrow \pi^-\pi^+p \dots$. The events are selected by the criterion that the $(\pi^-\pi^+)$ vertex is at least 35 mm downstream of the primary vertex.

Fig. 2.9 : 3π effective mass resolution in the reaction $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$ at 25 and 40 GeV/c for the 3C and 4C fit.

Fig. 2.10 : Boundaries of the apparatus in the π^-p centre-of-mass system at 40 GeV/c. The trigger selects events with a

proton in the proton window. In the further analysis we select events with two or three pions in the rightmost corner (magnet acceptance) and one or zero pions in the front chambers.

Fig. 2.11 : Scheme of the information flow from the experiment to the physical results.

Table 2.1

Summary of resolution and efficiency of the different parts of the CERN-IHEP Boson Spectrometer

Part of experiment	Experimental technique	Resolution *)	Efficiency
Beam	Scintillation hodoscopes	$\Delta p/p = \pm 0.3\%$	$(96 \pm 2)\%$
		$\Delta\theta = 0.5 \text{ mrad}$	
Proton telescope	Wire chambers	$\Delta\theta[\text{mrad}] = \frac{0.9}{p[\text{GeV}/c]} \cdot \beta$	$(99.3 \pm 0.3)\%$
	Time of flight (T2-R3)	$\Delta p[\text{MeV}/c] = \frac{E^2[\text{MeV}^2] \cdot p[\text{MeV}/c]}{25}$	
	Range telescope (P0-P15)	See Fig. 2.2a	100%
Magnetic spectrometer	Wire chambers + analysing magnet	$\Delta\theta[\text{mrad}] = 0.5 + 35\theta^2[\text{rad}]$	See Fig. 2.4
		$\Delta p/p[\%] = 0.13 \cdot p[\text{GeV}/c]$	

*) All errors refer to one standard deviation.



Fig. 2.2

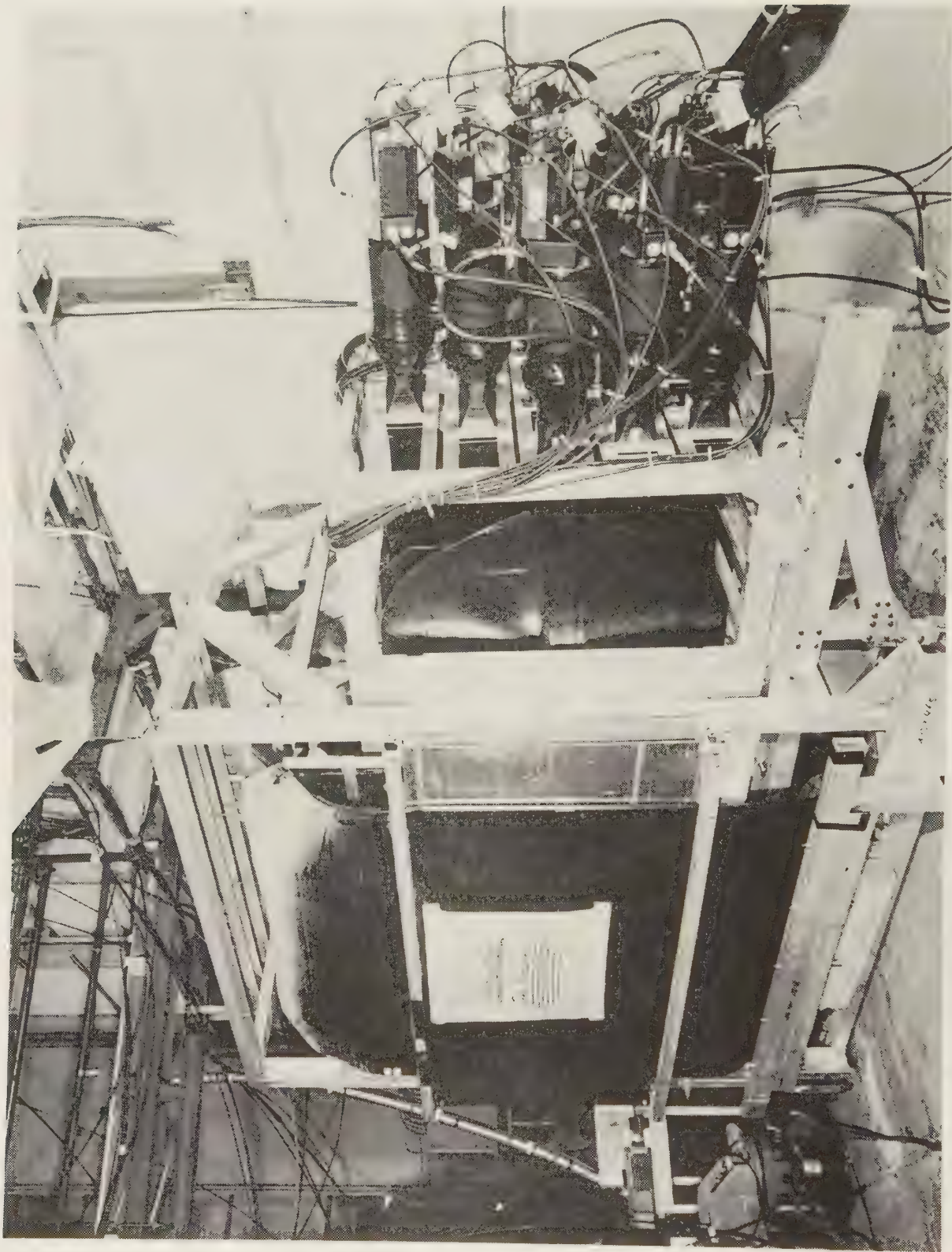


Fig. 2.3

PROTON MOMENTUM RESOLUTION

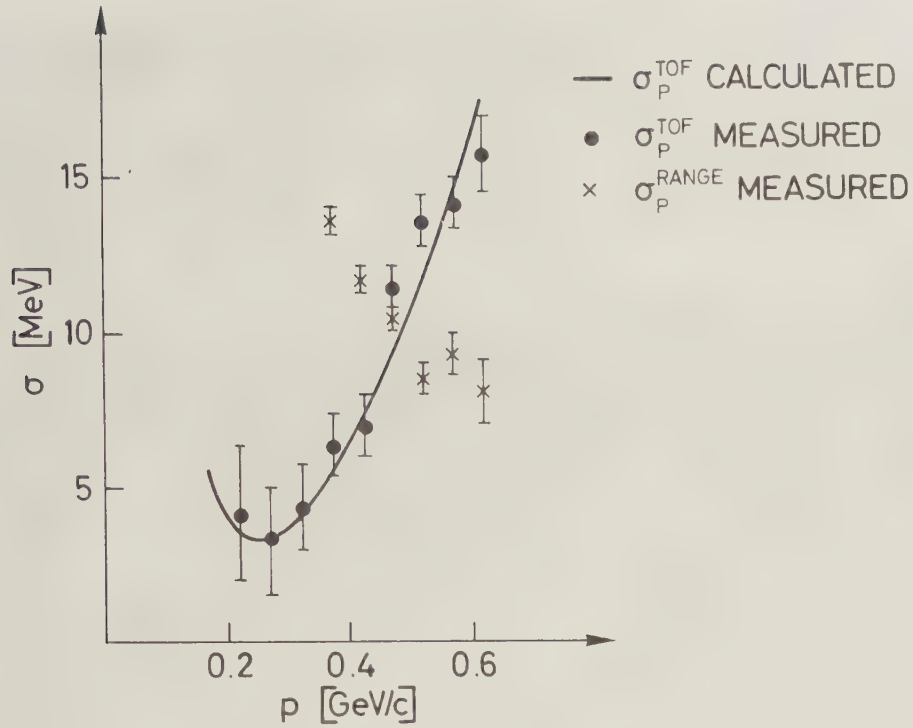


Fig. 2.4

PROTON ANGULAR RESOLUTION

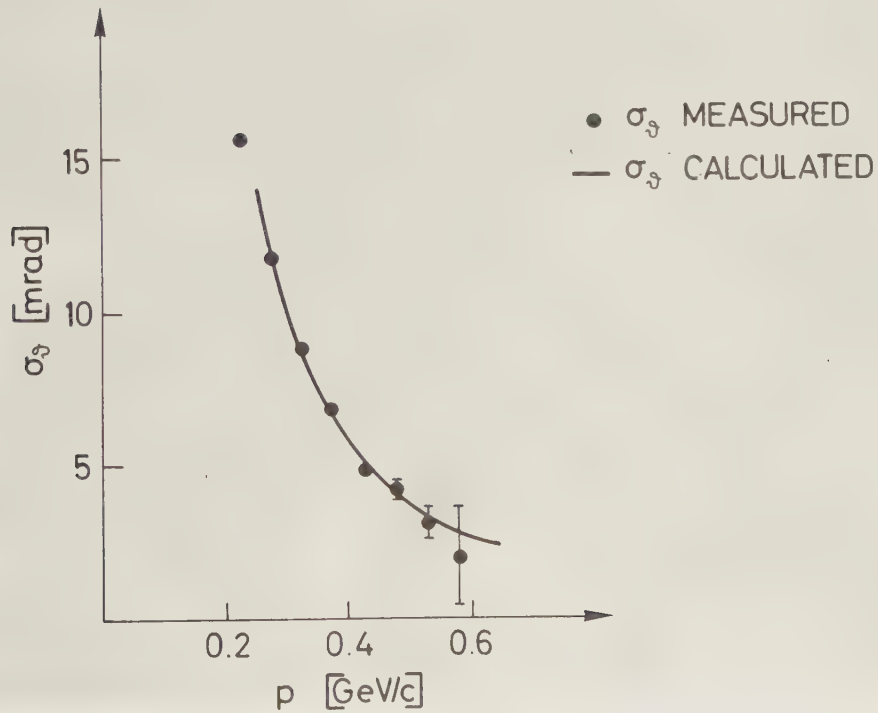


Fig. 2.5

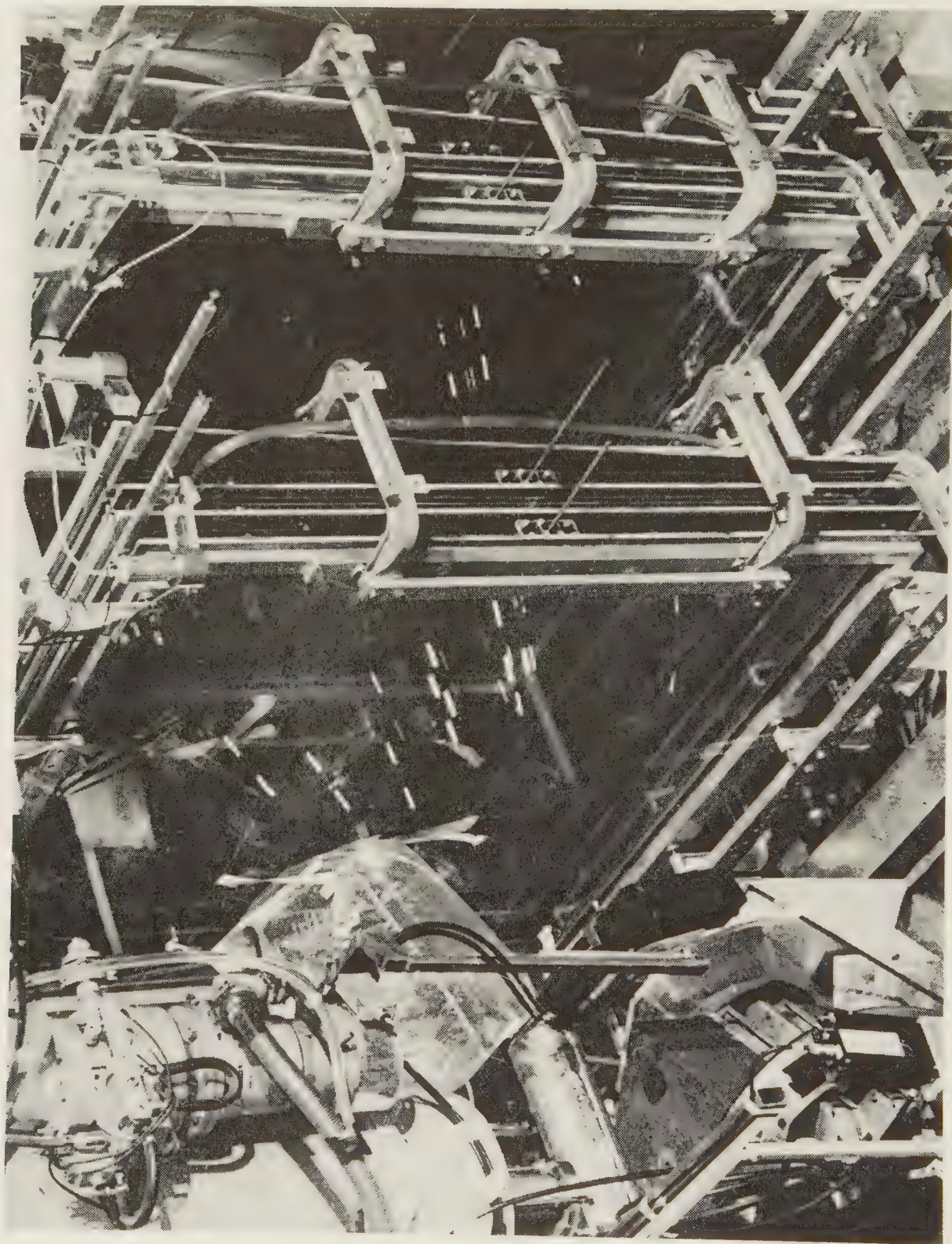
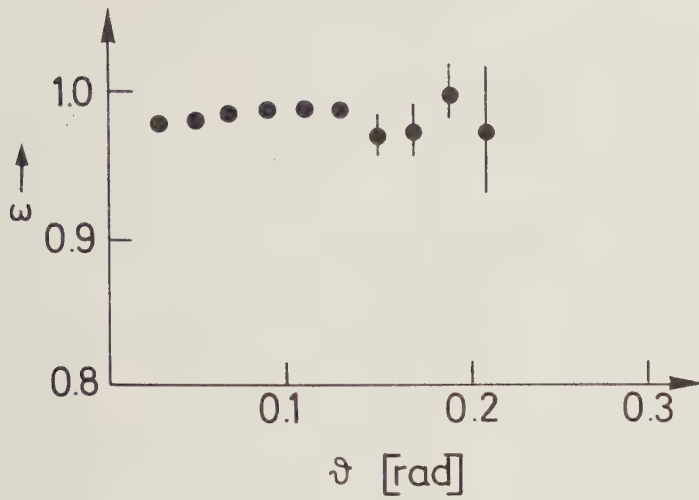


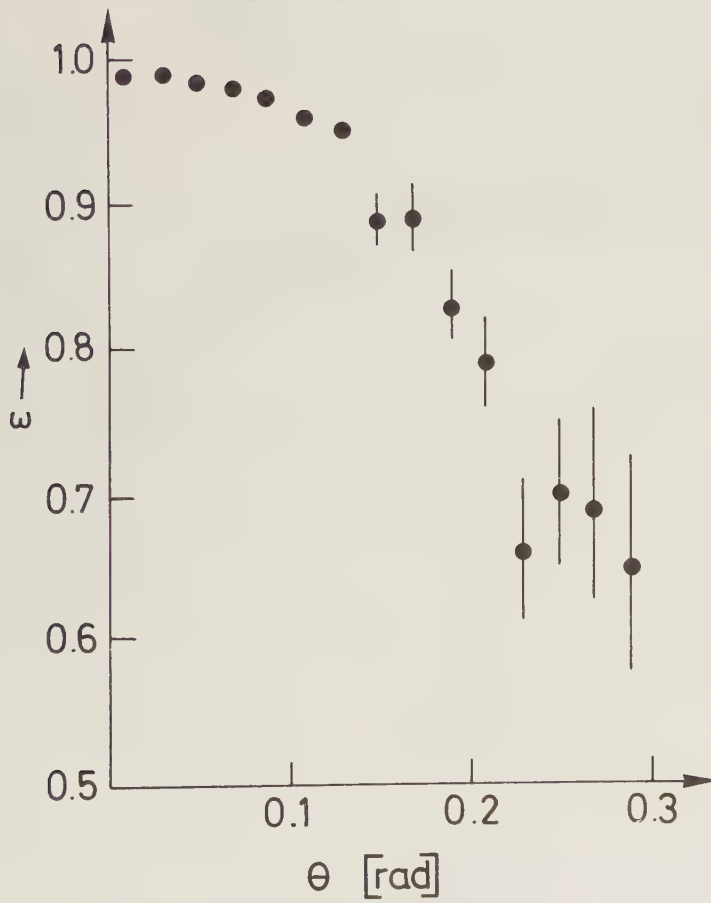
Fig. 2.6

EFFICIENCY OF THE CHAMBERS IN FRONT OF THE MAGNET



a)

EFFICIENCY OF THE CHAMBERS BEHIND THE MAGNET



b)

Fig. 2.7

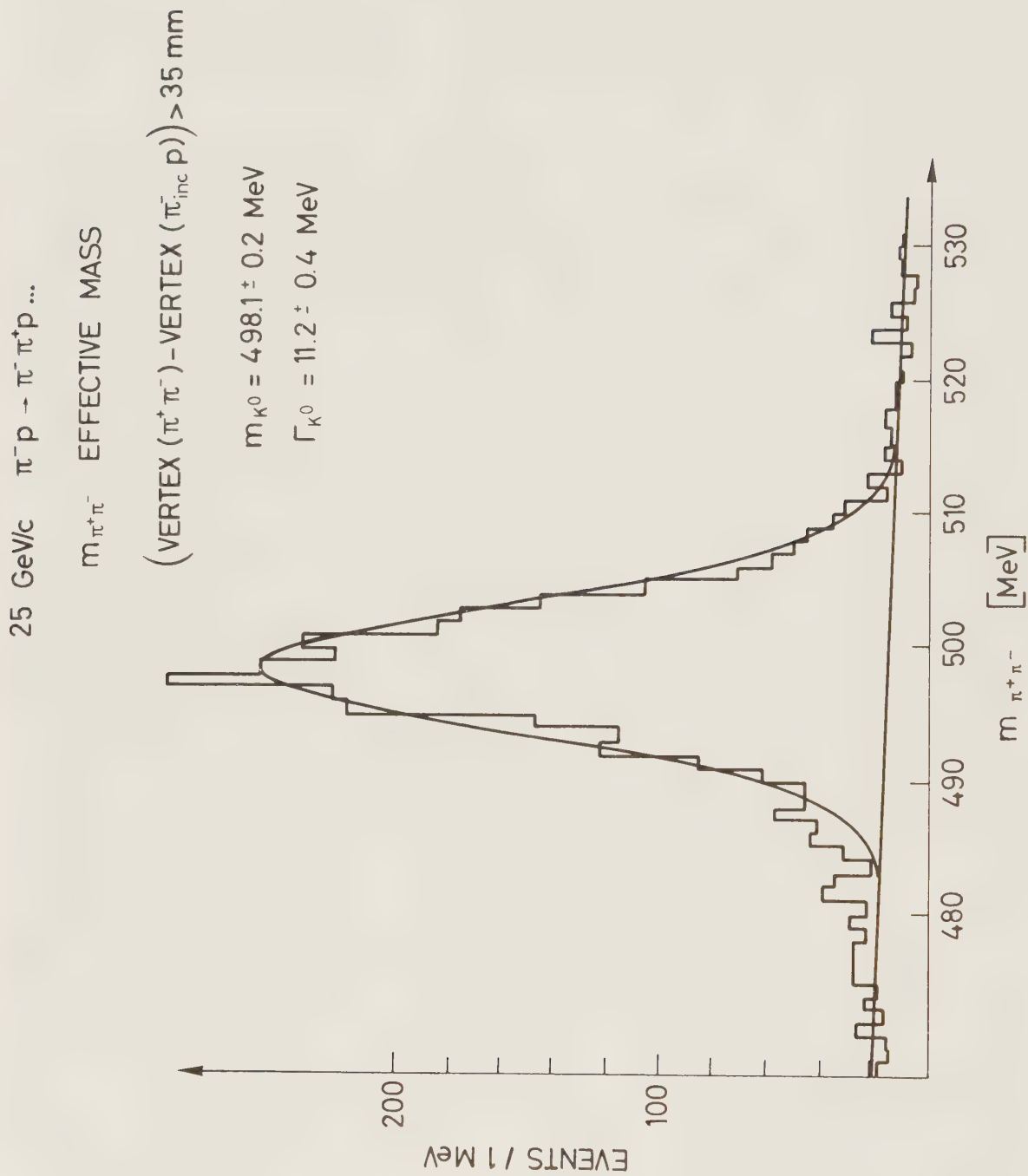


Fig. 2.8

3π - EFFECTIVE MASS RESOLUTION $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$

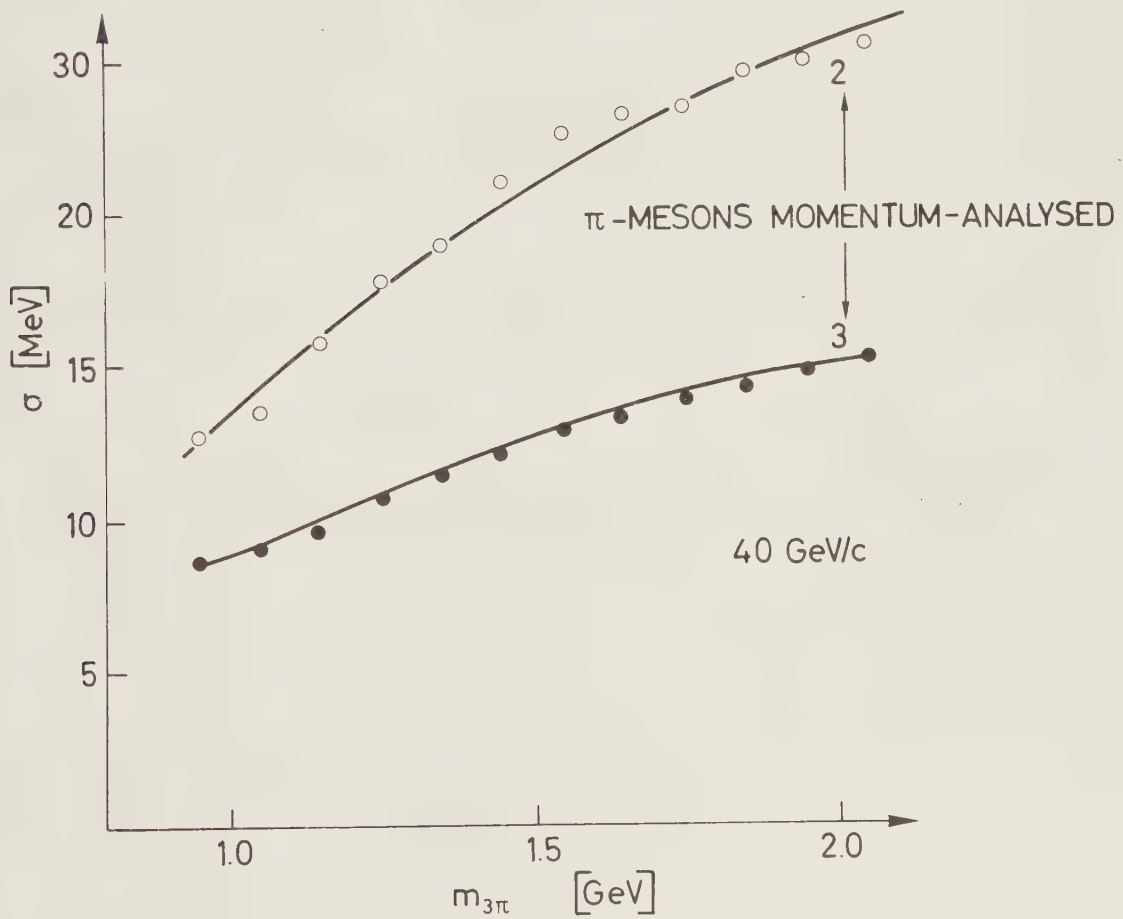
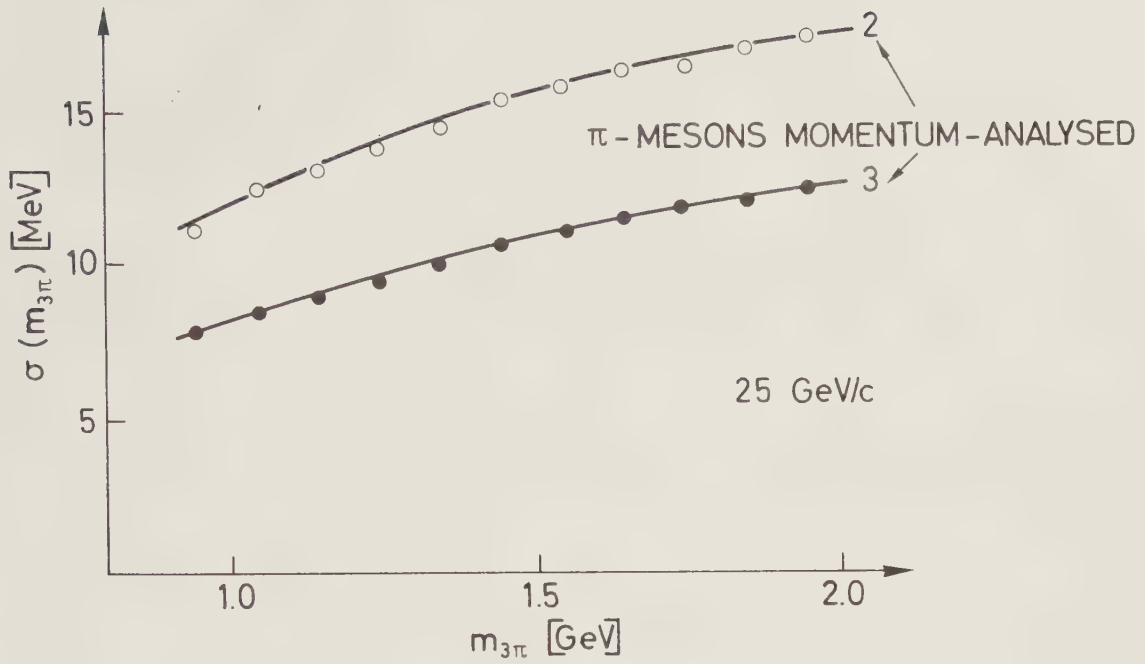


Fig. 2.9

CIBS 1971

Boundaries for the CM - acceptance for
 $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ at 40 GeV/c

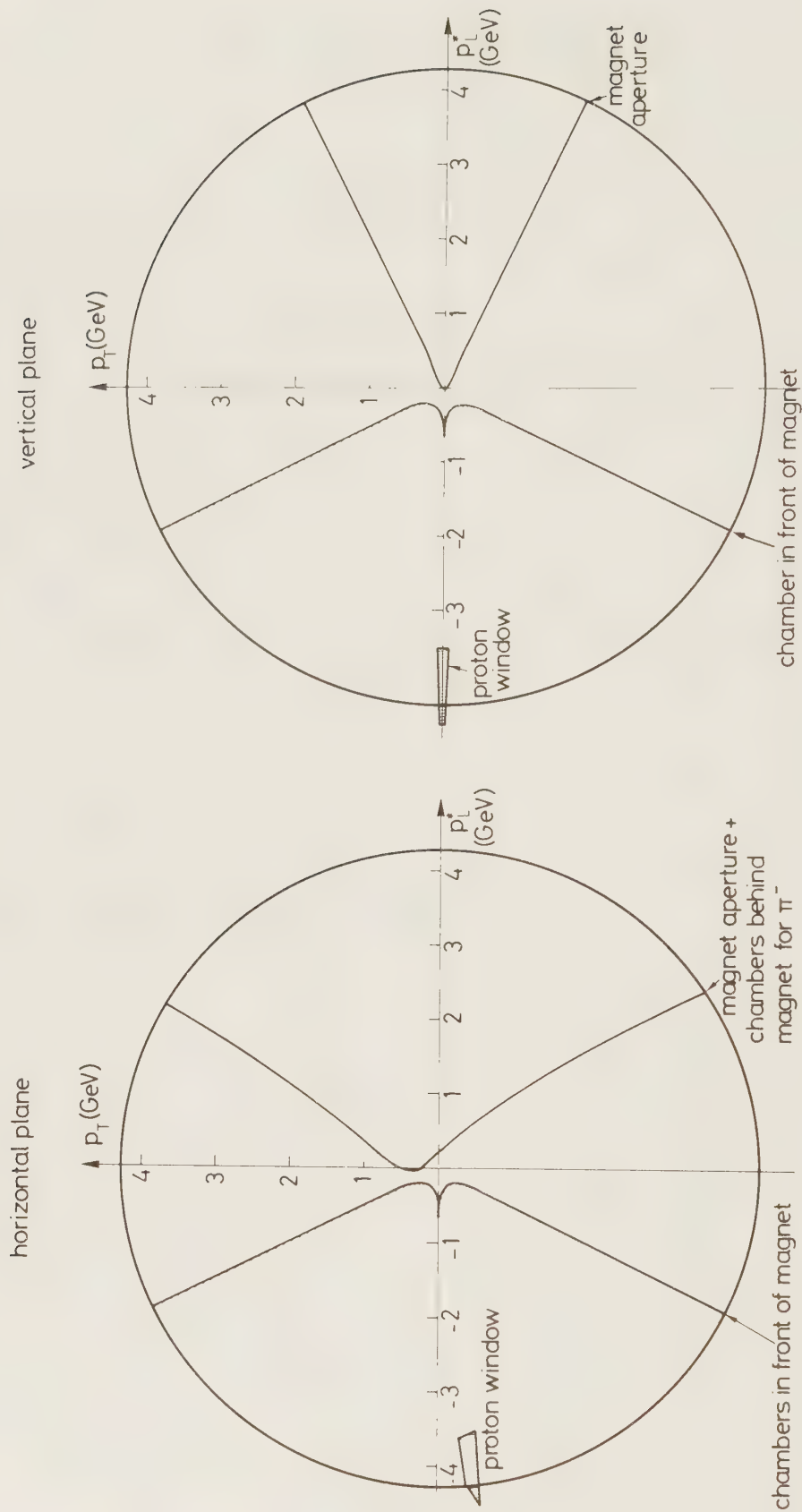


Fig. 2.10

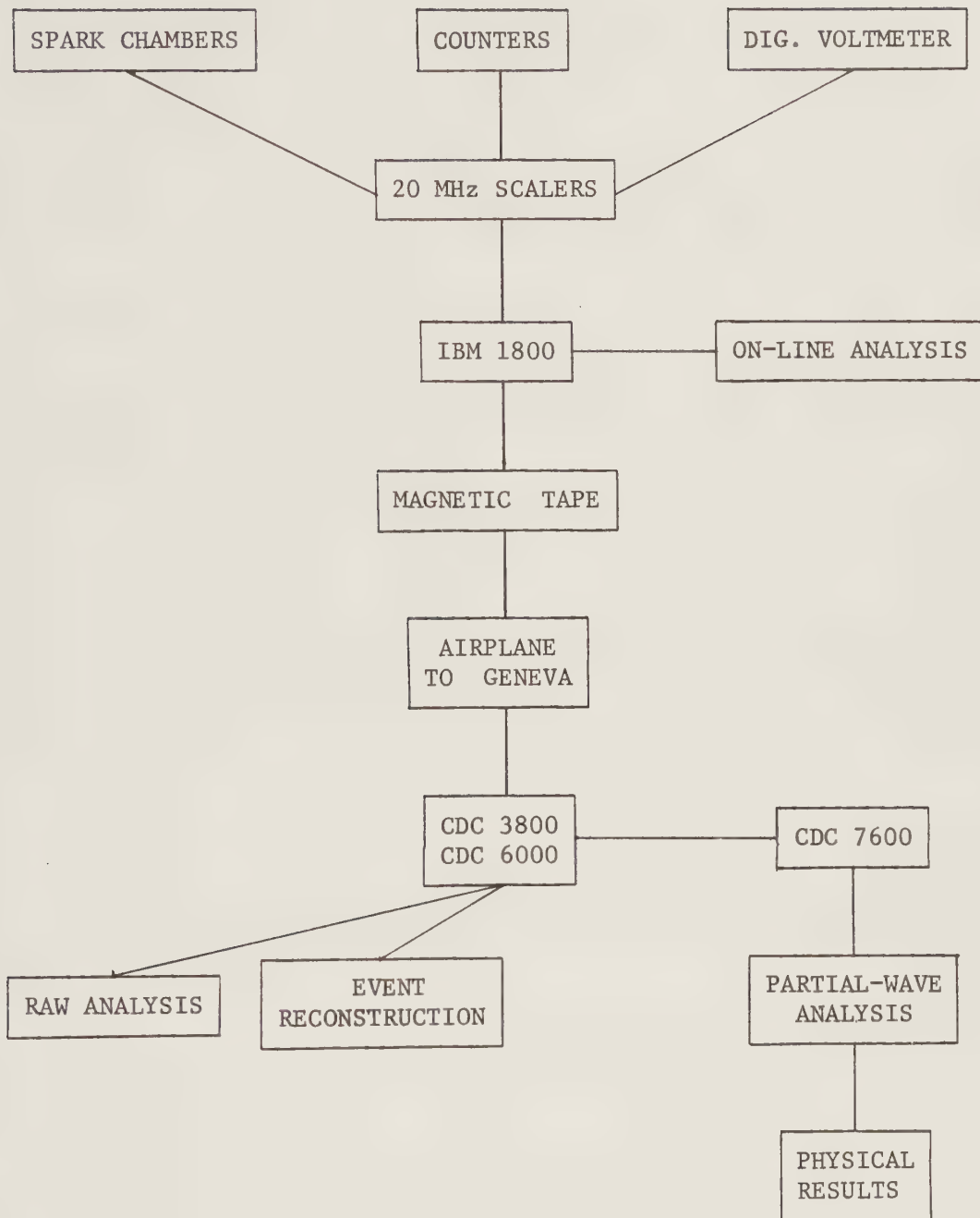


Fig. 2.11

CHAPTER 3

SELECTION OF EVENTS OF THE CHANNEL $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$

In this chapter the event selection is discussed, as well as the purity of the final data sample, and the correction factors for the cross-section determination. Table 3.1 summarizes the results of this chapter.

3.1 SELECTION OF EVENTS $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$

We use events from three running periods between April 1971 and October 1971. Table 3.2 lists the different experimental conditions, and the number of events acquired.

We give below the different selection criteria applied to the raw data.

- a) In roughly $4 \pm 2\%$ of the events the position and direction of the incoming beam track cannot be decoded from the beam hodoscopes. In most of these events two beam particles passed through the counters during the sensitive time of the hodoscope pattern units (50-70 nsec). Thus this cut rejects good events, and we have to take them into account when calculating the cross-section.
- b) In 5-12% of the events no track is found in the spark chambers of the recoil spectrometer. As the spark chamber efficiency is $99.3 \pm 0.3\%$ [see Chapter 2 (Section 2.3.2)], most of these events are due to accidental triggers.
- c) In 11% of the remaining events the recoil track found in chambers CH5 and CH6 does not intersect with the incident beam track inside the liquid hydrogen.
- d) 5% of the events are removed by demanding that the prolongation of the recoil track measured in the spark chambers has to hit P15, the anti-counter of the range telescope.
- e) In 25 or 15% of the events (depending on whether the anti-elastic telescope ElE2 was used in the trigger or not), three charged particle

tracks originating from the proton vertex (i.e. the intersection of the beam with the proton track) can be found in the forward spectrometer. To find these tracks in the presence of background tracks (on the average there are two of them), we calculate the point in space which minimizes the chisquare, the squared sum of the minimum distance between the tracks and this point divided by its error:

$$\chi^2 = \sum_i \left(\frac{d_i}{\sigma_i} \right)^2.$$

Whenever the χ^2 probability $P(\chi^2)$ is less than 0.001, we take the track which gives the biggest contribution to the χ^2 out of the sum and repeat the procedure until $P(\chi^2)$ is bigger than 0.001. We call the point from which the tracks originate the "track vertex". This procedure is repeated for the tracks rejected in the previous pass, until all track vertices are found. The tracks which originate from this vertex, which is within 15 mm of the proton vertex, are retained. Due to the inefficiency of the spark chambers and the dead-spots in the centre of the chambers, tracks might have been missed in the chambers in front of the magnet. We therefore accept also tracks that are only measured in the chambers behind the magnet, if they point in the vertical direction where the magnetic deflection is small, to the proton vertex (within 15 mm). We accept all events with three tracks found with these criteria. To check this selection we count how many events with three or more tracks (neglecting the selection criteria discussed above) can be found for which one combination of three tracks fulfils the constraints of momentum conservation, together with the incoming beam particle and the recoil proton. In this way we find 4% more events with two or three pions momentum-analysed than with the strict selection criterion. We thus conclude that we lose less than 4% of the good events by the track selection.

f) The hypothesis $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$ is fitted to the remaining events using the constraints of momentum and energy conservation^{3.1)}: a four-constrained fit (4C), if all three pions are momentum-analysed, a 3C, 2C, 1C fit if two, one or zero pions are momentum-analysed. About 40% of the events pass the kinematic fit with a χ^2 of less than 25. As in the further analysis we need the sign of the charge for all particles, we reject another

40% by rejecting 1C and 2C events, so that charge conservation allows us to calculate the sign of the charge of all particles.

g) As a last selection criterion we demand that the χ^2 probability of the fit exceeds 0.02 (see Fig. 3.1). Assuming an approximately flat probability distribution we find from Fig. 3.1 that not more than $2 \pm 2\%$ of the $\pi p \rightarrow \pi^- \pi^- \pi^+ p$ events are removed by this cut.

3.2 CONTAMINATIONS FROM OTHER CHANNELS

3.2.1 Incident particles other than pions

Only during part of the runs (October 1971) have the Čerenkov counters been working. Therefore 3-4% of the incoming particles are not pions, and we have no means of distinguishing reactions like $K^- p \rightarrow K^- \pi^- \pi^+ p$ from the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$.

3.2.2 Four-body reactions other than $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$

We can use the two independent measurements of the recoil momentum by time-of-flight p_{TOF} and range p_R to check how many of the recoil particles are actually protons, as we assume in the kinematical fit. The distribution of the difference $p_{\text{TOF}} - p_R$ shows a tail to positive values, as expected for protons undergoing nuclear interactions. From the tail of the distribution we estimate the probability that p_{TOF} is more than 10 MeV/c bigger than p_R . This number is compared in Fig. 3.2 to the probability that a proton suffers a nuclear interaction thereby losing more than 10 MeV/c calculated according to the tables of Measday and Serre^{3,2)}. Both curves agree, and we estimate that $97 \pm 3\%$ of the recoil particles are actually protons.

Next we estimate the contamination from the reactions $\pi^- p \rightarrow \pi^- K^- K^+ p$ and $\pi^- p \rightarrow \pi^- p p p$. As the momentum constraints are independent of the particle masses, the energy balance (i.e. the difference between outgoing and incoming energy) is the only way to distinguish these events from $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ events. Figure 3.3 shows the energy balance for all events with four outgoing tracks after the momentum conservation fit, assuming that the forward particles are pions. The unshaded histogram in Fig. 3.3 is the energy balance for events fitting also the energy-conservation fit for $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$. Their energy balance is centred

around zero with a standard deviation of ~ 3 MeV. The shaded histogram shows the energy balance for the events not fitting $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$. Assigning the pion mass to the two K mesons in the reaction $\pi^- p \rightarrow \pi^- K^+ K^- p$ shifts the energy balance by typically 15 MeV downwards (for the $\pi^- p \rightarrow \pi^- p p p$ events ~ 50 MeV), which is actually observed. Counting the events in the overlap region we estimate an upper limit of 1% for this contamination.

3.2.3 Reactions with missing neutrals

Due to the measurement errors on the momentum vectors of the charged particles, roughly 30% of the selected events also fit the hypothesis $\pi^- p \rightarrow \pi^- \pi^- \pi^+ \pi^0 p$ with a χ^2 probability of more than 2%. For these events, however, the longitudinal momentum p_L , and the transverse momentum p_T of the neutral particles is confined to the interval $0 \leq p_L \leq 2.5$ GeV/c, $|p_T| \leq 0.1$ GeV/c. To estimate the contamination of our data sample from events with missing neutral particles we calculate the missing transverse and longitudinal momenta (p_T^0, p_L^0) for all events with three charged particles in the forward spectrometer. In Fig. 3.4 we compare the transverse momentum squared $(p_T^0)^2$ for the events that fit the hypothesis $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$, and the events that do not fit this hypothesis, and have a longitudinal momentum p_L^0 of less than 2.5 GeV/c. Assuming that the smooth, hand-drawn curve that describes the distribution above 0.01 (GeV/c) 2 can be extrapolated to zero, we find an upper limit of 2% for this contamination.

3.3 CROSS-SECTION NORMALIZATION

In this section we discuss the counting of the incident beam flux, and all correction factors for the cross-section normalization which have not yet been discussed in the previous chapters.

For the cross-section evaluation we have selected runs (a run contains about 15,000 raw triggers), during which the incident beam had no structure and was well centred on the target, the background conditions were good (i.e. few accidental triggers), and the experiment was performing well (e.g. high spark-chamber efficiency, etc.). For the selected runs it has been checked that the different rejection rates and the

percentage of final events were constant within their statistical error. For these groups of runs the number of incident beam particles was counted and the different correction factors determined to obtain the cross-section for different momentum-transfer intervals Δt and 3π mass intervals $\Delta m_{3\pi}$. To determine the cross-section for the production of a resonance in a certain bin Δt and $\Delta m_{3\pi}$, we can use the events from all runs to determine with good precision the ratio of resonance to total number of events (see Chapter 6), and multiply it with the cross-section for this $\Delta m_{3\pi}$ and Δt bin.

3.3.1 Counting of the beam flux and number of target protons

The number of incident particles detected in the beam telescope T1T2T0 was counted by two 20 MHz scalers. At an average intensity of 4×10^5 particles/sec, we estimate $2 \pm 1\%$ counting losses due to the dead-time of the scalers.

To estimate the number of beam particles counted in the beam telescope but passing outside the liquid hydrogen target, we compare the distribution of the beam for the selected events with the beam distribution for calibration runs where the beam telescope was the only trigger requirement. From this we find that $2 \pm 5\%$ of the counted tracks miss the target.

From the $\pi^- p$ total cross-section of ~ 25 mb above 20 GeV/c^{3.3}), we calculate an average beam attenuation of $2 \pm 0.5\%$ for our 43 cm long liquid hydrogen target. The uncertainty comes from $\pi^- p$ elastic events, where most of the π 's are produced with practically the same momentum as the beam.

The density of liquid hydrogen ($\rho = 0.0708$ g/cm³), needed to calculate the number of target protons, is estimated to be known within $\pm 1\%$. As we determine the vertex position along the beam with good precision (1 mm), we have no error on the target length ℓ , if we select events produced well inside the liquid hydrogen. From the atomic weight of hydrogen $A_H = 1.01$ g, and the Avogadro number $N_A = 6.02 \times 10^{23}$, we obtain the number of target protons per unit surface:

$$N_p = (N_A \rho \ell) / A_H .$$

For the remaining correction factors we distinguish between effects that are approximately independent of the physical variables of the individual event and thus give an over-all correction factor, and effects that depend on the physical variables.

3.3.2 Over-all correction factors

These are for the trigger inefficiency of $3 \pm 1\%$ due to the proportional wire chamber, the spark chamber inefficiency of $0.7 \pm 0.3\%$, the number of recoil protons undergoing nuclear interactions before producing a trigger signal in the R3 counter of $2 \pm 1\%$, as well as the cuts for event selection, and the contamination from other reactions discussed in Sections 3.1 and 3.2.

3.3.3 Correction factors that depend on physical variables

Here we want to determine how many of the $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ events (at a fixed set of production variables for the 3π system) are lost due to effects that depend on the physical variables that describe the reaction. The problem is that we have to know the physical distribution function in order to determine this acceptance factor. On the other hand, we need the acceptance function to find the physical distribution from the measured data. As we do not have a functional expression for the acceptance function but only a recipe of how to calculate the acceptance $A(\vec{\tau})$ for any event with the variables $\vec{\tau}$, we use the following method to determine the acceptance factor.

We first develop the unknown physical distribution function $W(\vec{\tau})$, which describes the decay of the 3π system into the functions^{*)} $f_j(\vec{\tau})$, with the unknown parameters C_j

$$W(\vec{\tau}) = \sum_j C_j f_j(\vec{\tau}).$$

We then calculate by Monte Carlo integration the acceptance integrals $a_j = \int d\vec{\tau} A(\vec{\tau}) \cdot f_j(\vec{\tau})$. Next we determine the parameters C_j , which contain all physical information, by the maximum likelihood method from the data.

*) See Chapter 5 for the choice of the functions.

Writing explicitly the logarithm of the likelihood function

$$\begin{aligned}\ln \mathcal{L} &= \ln \left(\prod_{i=1}^N A(\vec{\tau}_i) W(\vec{\tau}_i) \right) - N \int d\vec{\tau} A(\vec{\tau}) W(\vec{\tau}) \\ &= \sum_{i=1}^N \ln W(\vec{\tau}_i) + \sum_{i=1}^N \ln A(\vec{\tau}_i) - N \sum_j C_j a_j\end{aligned}$$

$\vec{\tau}_i, \dots$ variables of the N measured events

we see, that the acceptance function $A(\vec{\tau})$ enters only through the integrals a_j into the likelihood function. From the best estimates of the parameters C_j we then obtain the acceptance factor:

$$\sum_j C_j a_j / \sum_j C_j \int d\vec{\tau} f_j(\vec{\tau}).$$

In the following we discuss the different effects we take into account when calculating the acceptance integrals.

To calculate the absorption of the forward produced π mesons in the liquid hydrogen we parametrize the total π^+p cross-section as a function of the π momentum $24(1 + 1/p [\text{GeV}/c])\text{mb}$, and obtain for the absorption per cm of liquid hydrogen $(1 + 1/p [\text{GeV}/c]) \times 10^{-3}$. For the Monte Carlo integration we generate $\pi p \rightarrow 3\pi$ events, originating from the centre of the target, and calculate the path of every π meson in the hydrogen, and multiply it with the formula above to obtain the probability that one of the π mesons is absorbed. The average effect is about 8% and we correct for it within $\pm 2\%$.

A smaller effect is the pion decay, which is on the average about 1.5%. We estimate that we correct for this effect within $\pm 1\%$. The correction for the finite solid angle of the apparatus consists of two terms.

First there is a factor of roughly 24, which depends only slightly on the 3π effective mass and momentum transfer, due to the solid angle of the recoil spectrometer. As beam and target particles are unpolarized this cut does not introduce any bias for the measurement of the decay of the 3π system.

The finite size of the forward spectrometer, including the dead spots in the centre of the chambers, is corrected for in the Monte Carlo integration by cutting out events with more than two of the forward particles missing the chambers behind the magnet or one of the particles missing the chambers in front of the magnet. These are the same selection criteria as are used for the measured events. Figure 3.5 shows how big this correction is for the data at 25 and 40 GeV/c. We estimate that our correction is good to $\pm 15\%$. This error is mainly due to the cut-off of the dead-spots in the centre of the chambers, which is not very well defined.

In Chapter 2 (Section 2.4) we discussed how we obtained the spark-chamber efficiencies from the $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ events. In the Monte Carlo integration we correct for it by weighting every generated event by the probability that we detect all three pions in the chambers. We estimate that we correct this effect, which is of the order of 5 to 10%, to better than $\pm 20\%$.

Adding all systematic errors quadratically we estimate a total systematic error of $\pm 10\%$ at a 3π mass of 1 GeV (slightly increasing with increasing mass).

REFERENCES

- 3.1) B. Ronne, Kinematical analysis of bubble chamber pictures *in* Proc. Herceg-Novi Summer School, 1964, CERN 64-13 (1964).
- 3.2) D.F. Measday and C. Richard-Serre, Loss of protons by nuclear interactions in various materials, CERN 69-17 (1969).
- 3.3) E. Bracci, J.P. Droulez, E. Flaminio, J.D. Hansen and D.R. Morrison, Compilation of π^- and π^+ induced reactions, CERN/HERA 72-1 (1972).

TABLE CAPTIONS

- Table 3.1 : Summary of cross-section correction factors.
- Table 3.2 : Statistics on the events used for the analysis.

FIGURE CAPTIONS

- Fig. 3.1 : χ^2 probability distribution of the kinematical fit to the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$: (a) for the 4C fit and (b) for the 3C fit.
- Fig. 3.2 : Measured and calculated probability that a proton suffers a nuclear interaction, thereby losing more than 10 MeV/c, versus proton momentum.
- Fig. 3.3 : Energy balance after momentum conservation fit for the four-body events assuming three-forward pions: shaded events not fitting the hypothesis $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$, blank events fitting the hypothesis $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$.
- Fig. 3.4 : Distribution of the transverse π^0 momentum squared for: shaded events fitting the hypothesis $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$, blank events not fitting the hypothesis $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$.
- Fig. 3.5 : Acceptance of the forward spectrometer for the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ at 25 and 40 GeV/c.

Table 3.1

Cross-section correction factors

Effect	Correction factor	Part of experiment
Beam hodoscopes	0.96 ± 0.02	Beam and target
Counting of beam particles	0.98 ± 0.01	
Particles missing target	$0.98 \begin{matrix} + 0.02 \\ - 0.05 \end{matrix}$	
Particle identification	0.97 ± 0.03	
Beam absorption in target	0.98 ± 0.006	
Hydrogen density	1.00 ± 0.01	
Spark chambers	0.997 ± 0.003	Proton telescope and trigger
Nuclear interactions	0.98 ± 0.01	
Trigger inefficiency	0.97 ± 0.03	
Proton identification	0.97 ± 0.03	
Solid angle	See Sect. 3.3.3	
Solid angle	See Sect. 3.3.3	Magnetic spectrometer
Spark chambers	See Sect. 3.3.3	
3 track selection	$1.00 \begin{matrix} + 0.00 \\ - 0.04 \end{matrix}$	
π absorption	0.92 ± 0.02	
π decay	0.985 ± 0.01	
2 or 3 π 's momentum analysed	See Sect. 3.3.3	Event selection
χ^2 cut	0.98 ± 0.02	
KK π or $p\bar{p}\pi$ events	$1.00 \begin{matrix} + 0.01 \\ - 0.00 \end{matrix}$	
Events with missing neutrals	$1.00 \begin{matrix} + 0.02 \\ - 0.00 \end{matrix}$	

Table 3.2
Statistics on the events used for the analysis

Running period	April 1971	June 1971	June 1971	June 1971	October 1971
Incident momentum [GeV/c]	40	25	25	40	40
Momentum transfer $[(\text{GeV}/c)^2]$	0.17-0.33	0.17-0.33	0.10-0.33	0.10-0.33	0.04-0.33
Number of raw triggers	549,800	446,800	410,050	288,550	349,000
Number of accepted events	15,200	15,800	15,550	10,000	15,030

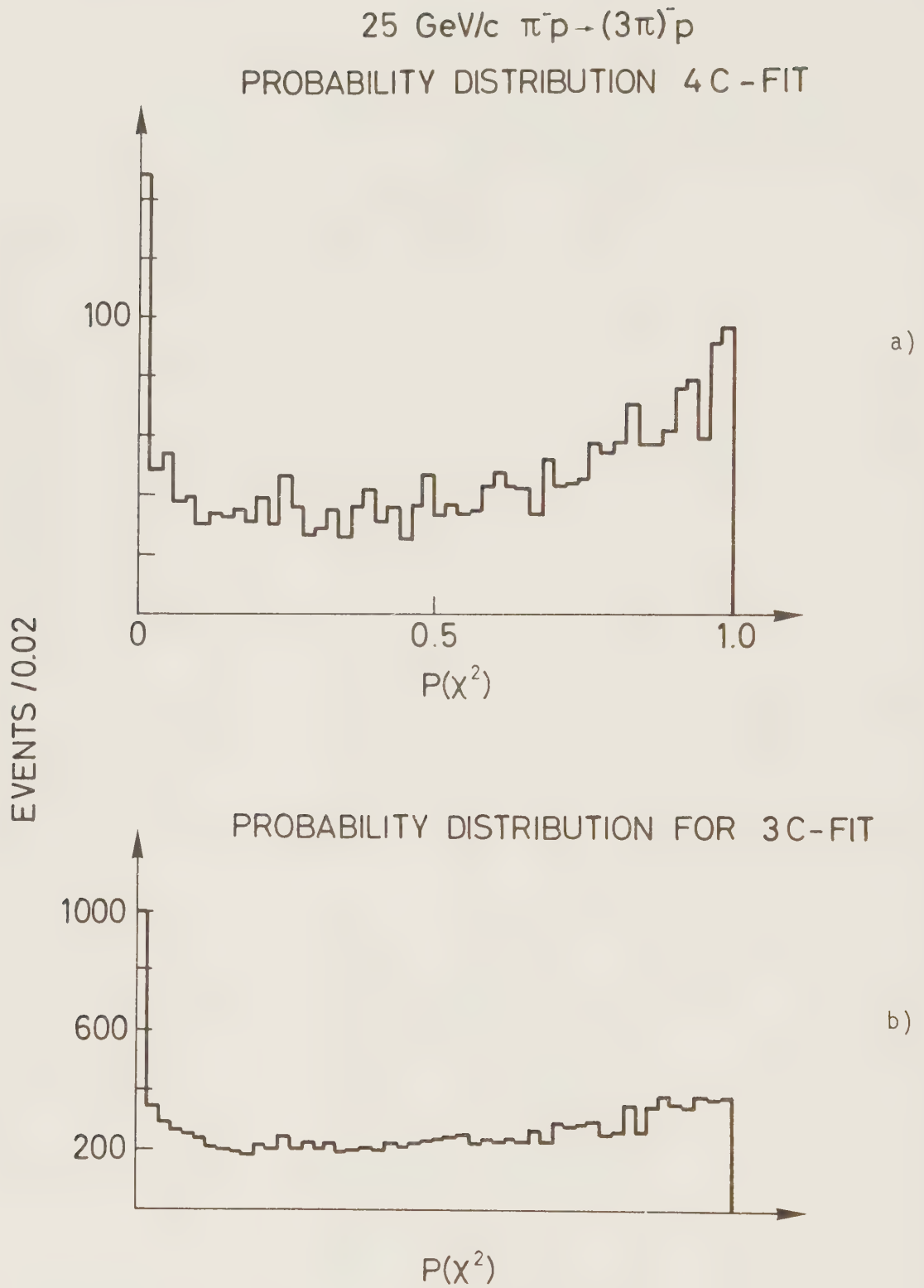


Fig. 3.1

PERCENTAGE OF PROTONS SUFFERING A NUCLEAR
INTERACTION

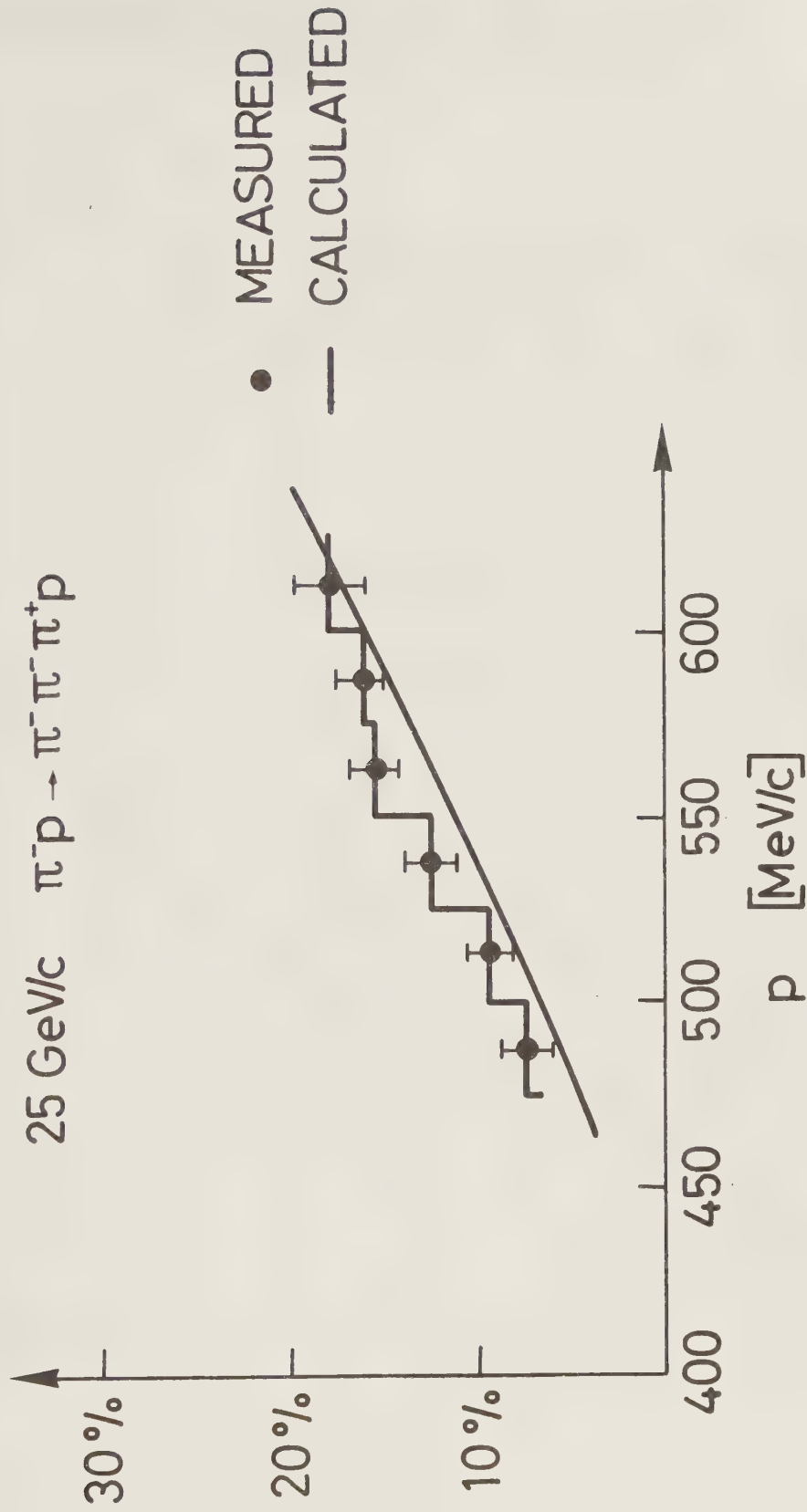


Fig. 3.2

ENERGY BALANCE AFTER MOMENTUM CONSERVATION
FIT FOR 40 GeV/c π^-p (OCT.71)

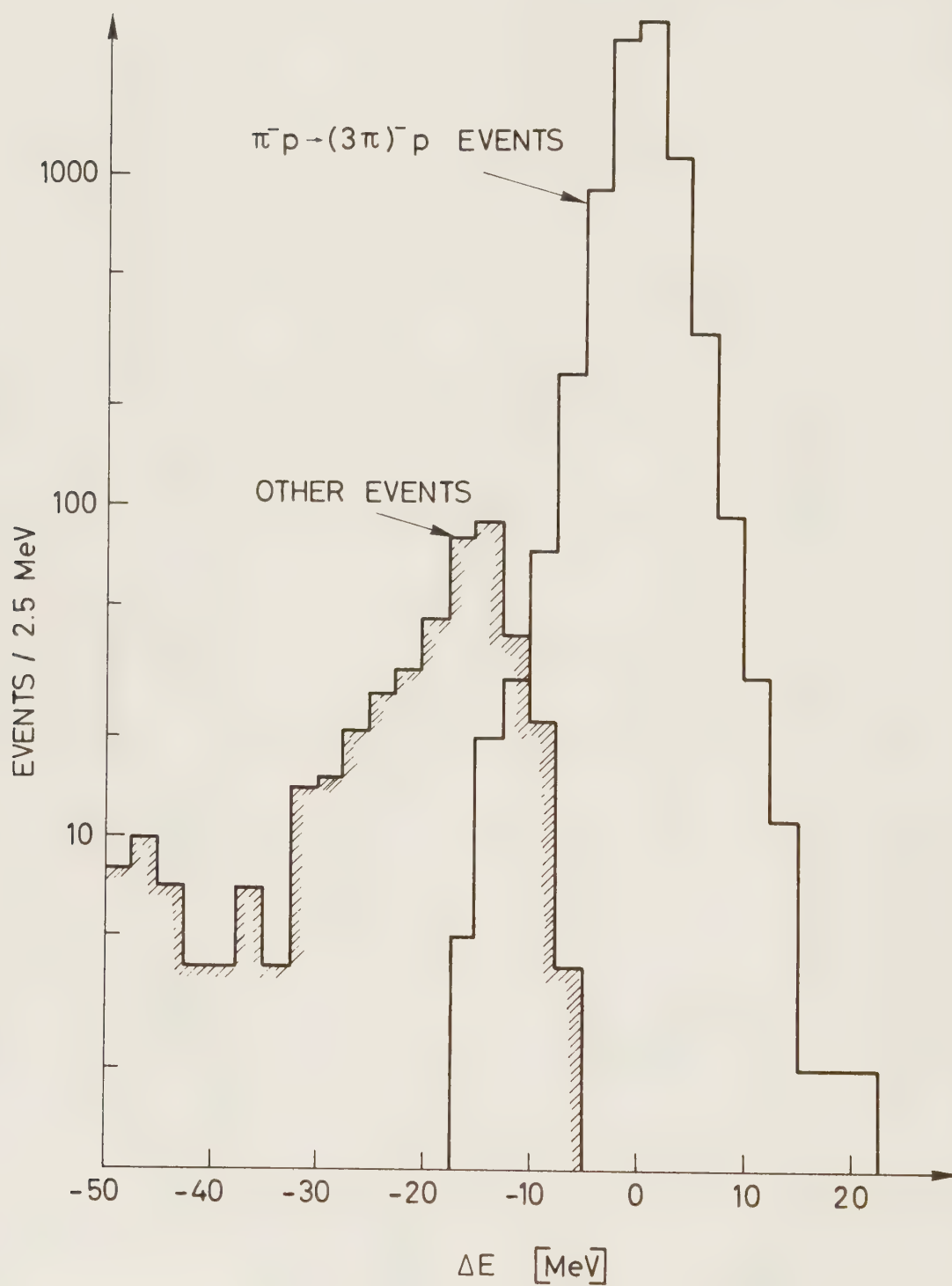


Fig. 3.3

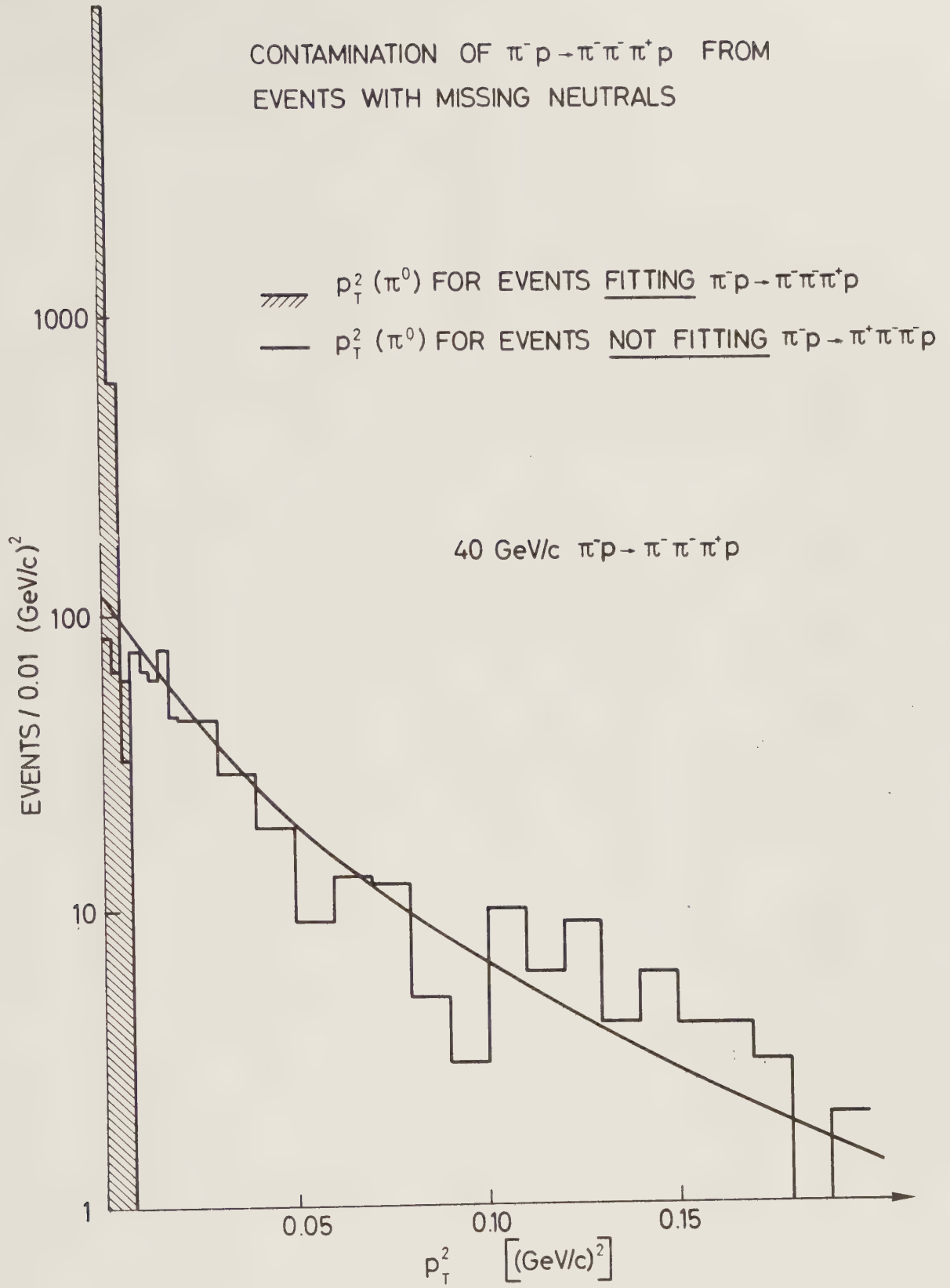


Fig. 3.4

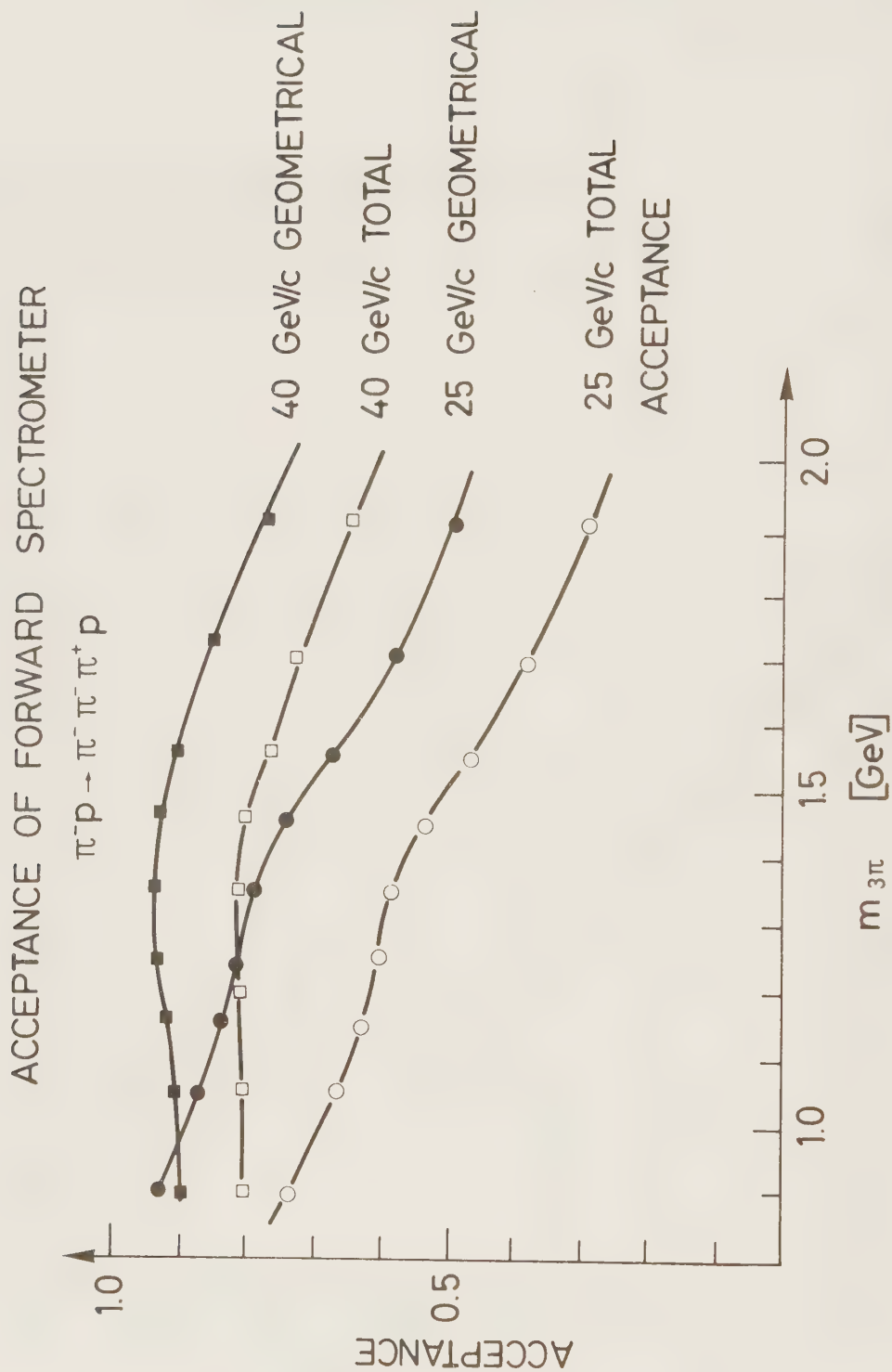


Fig. 3.5

CHAPTER 4

DISCUSSION OF THE MAIN FEATURES OF THE DATA

In this chapter some of the features of the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ at high energies are discussed. As the 25 GeV/c data after correction for all acceptances are very similar to the 40 GeV/c data, we will show histograms of the latter sample only where the corrections are smaller.

4.1 EFFECTIVE MASS PLOTS

Figure 4.1 shows the corrected 3π effective mass spectra for the two momentum transfer intervals $t = 0.04-0.17$ (GeV/c)² and $t = 0.17-0.33$ (GeV/c)², for the data measured at 40 GeV. In the low t spectrum we find two broad bumps with the parameters of^{4.1}):

A1 meson, $m = 1150$ MeV, $\Gamma = 350$ MeV ($J^P = 1^+$),

A3 meson, $m = 1650$ MeV, $\Gamma = 300$ MeV ($J^P = 2^-$).

For the high t spectrum we find, in addition, strong production of the

A2 meson, $m = 1300$ MeV, $\Gamma = 100$ MeV ($J^P = 2^+$).

Here we make no effort to determine mass, width, and production cross-section for these mesons from the mass spectra, since we think we can do this only by separating the different spin-parity (J^P) states.

The $\pi^+ \pi^-$ effective mass spectrum (Fig. 4.2) shows the ρ^0 meson ($m = 765$ MeV, $\Gamma = 140$ MeV) and the f^0 meson ($m = 1260$ MeV, $\Gamma = 165$ MeV). Cutting out events with at least one $\pi^+ \pi^-$ effective mass combination in the ρ^0 meson region ($665 < m_{\pi^+ \pi^-} < 865$ MeV) shows that A1 and A2 mesons decay mainly via $\rho^0 \pi^-$ (shaded area in Fig. 4.1) contrary to the A3. No structure is seen in the $\pi^- \pi^-$ effective mass spectrum (Fig. 4.2b). The broad bump near 0.7 GeV is a reflection of the ρ^0 signal in the $\pi^+ \pi^-$ mass spectrum. Figures 4.3a and 4.3b are the $p\pi^+$ and $p\pi^-$ effective mass spectra, respectively. Since our apparatus selects protons, produced

backwards in the centre-of-mass frame, and pions, produced forwards (Fig. 2.10), the low $p\pi$ effective masses are strongly suppressed. The overlap of the A mesons with N^* resonances, which very much complicates the analysis of the A mesons at lower energies, is thus absent in our data sample.

4.2 CROSS-SECTION FOR THE REACTION $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ VERSUS 3π EFFECTIVE MASS, INCIDENT BEAM MOMENTUM, AND MOMENTUM TRANSFER

In order to compare the total cross-section for different 3π mass bins with data from other experiments, we first have to determine the shape of the differential cross-section to correct for the unobserved momentum transfer interval.

Figure 4.4 shows the slope parameter b obtained from fitting the expression $\exp(-b \cdot t)$ for different 3π mass bins to the differential cross-sections at 25 and 40 GeV/c. We observe a steep slope of the order of $15 (\text{GeV}/c)^{-2}$ at 3π masses around 0.9 GeV, smoothly decreasing towards $\sim 6 (\text{GeV}/c)^{-2}$ for $m_{3\pi}$ above 1.5 GeV. Within the errors the slope values at 25 and 40 GeV/c are the same. Assuming that the $\exp(-b \cdot t)$ dependence describes the differential cross-section in the full t region, we calculate the integrated cross-sections displayed in Table 4.1. The statistical error also contains the statistical uncertainty of the extrapolation of the t dependence. The systematic error contains all effects discussed in Chapter 3. Below a 3π mass of 2 GeV, the cross-sections at 40 GeV/c are equal to those at 25 GeV/c within their errors.

In Fig. 4.5 we compare our cross-sections with bubble-chamber data at lower energies^{4.2,4.3}). Below an incident momentum of 10 GeV/c many of the events with low 3π masses show a strong N^* signal in the $p\pi$ effective masses. Above 10 GeV/c, where this effect becomes smaller, the 3π cross-sections decrease only slowly with increasing energy. Fitting the expression $\sigma = a \cdot p^n$ to the data with the incident momenta p between 11 and 40 GeV/c, we find for low 3π masses a value of $n = -0.3 \pm 0.1$ (see Fig. 4.6 and Table 4.2), a value close to the exponent $n = -0.26 \pm 0.01$ found for $\pi^- p$ elastic scattering between 3 and 40 GeV/c^{4.4}).

As the value of n is believed to characterize the production mechanism^{4,5)}, we conclude that the same production mechanism is at work for πp elastic scattering and for the production of low mass 3π systems. For 3π masses above 1.4 GeV we find $n = -0.45$.

Summarizing, we have found that:

- i) there is a strong correlation between slopes of differential cross-section and 3π effective mass for the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$;
- ii) for 3π masses below 1.4 GeV the production cross-section decreases with the same power n as the $\pi^- p$ elastic cross-section;
- iii) there is strong production of A1, A2, and A3 mesons at both 25 and 40 GeV/c incident momentum.

In order to obtain detailed information about production and decay properties of the A mesons we separate them according to their different spin-parity values by a "Partial-Wave Method" as described in the next chapter.

REFERENCES AND FOOTNOTES

- 4.1) At this stage the properties of the particles are taken from Particle properties, Review of particle properties, Phys. Letters, April 1972.
- 4.2) The presentation of the cross-section $\sigma(\pi^- p \rightarrow \pi^- \pi^- \pi^+ p)$ versus $m_{3\pi}$ and incident momentum is as in the paper by K. Paler, Nuclear Phys. B18, 211 (1970).
- 4.3) The bubble chamber events come from the following sources:
 - 5 GeV/c: University of Illinois,
 - 7 GeV/c: Toronto-Wisconsin collaboration,
 - 7.5 GeV/c: University of Illinois,
 - 11 GeV/c: Genova-Hambourg-Milano-Saclay collaboration,
 - 13 GeV/c: Harvard University,
 - 16 GeV/c: Aachen-Berlin-Bonn-CERN-Heidelberg collaboration
 - 18.5 GeV/c: Notre-Dame University,
 - 20 GeV/c: Harvard University,
 - 25 GeV/c: University of Wisconsin.
- 4.4) Yu.M. Antipov, G. Ascoli, R. Busnello, G. Damgaard, M.N. Kienzle-Focacci, W. Kienzle, R. Klanner, L.G. Landsberg, A.A. Lebedev, C. Lechanoine, P. Lecomte, M. Martin, V. Roinishvili, R.D. Sard, A. Weitsch and F.A. Yotch, Elastic scattering of $\pi^- p$, $K^- p$, and $\bar{p} p$ at 25 and 40 GeV/c, Submitted to Phys. Letters, December 1972.
- 4.5) D.R.O. Morrison, Phys. Letters 22, 528 (1966); 25 B, 238 (1967).

TABLE CAPTIONS

- Table 4.1 : Cross-sections for the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ versus 3π effective mass at 25 and 40 GeV/c.
- Table 4.2 : Exponent n of the energy dependence of the $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ cross-section between 11 and 40 GeV/c ($\sigma \propto p_{inc}^{+n}$).

FIGURE CAPTIONS

- Fig. 4.1 : 40 GeV/c 3π effective mass spectra corrected for the acceptance and efficiency of the apparatus:
- a) All events with $0.04 < t < 0.17$ (GeV/c)².
Shaded events with one $\pi^+\pi^-$ mass combination in the ρ^0 meson region ($0.665 < m_{\pi^+\pi^-} < 0.865$).
 - b) All events with $0.17 < t < 0.33$ (GeV/c)².
Shaded events with one $\pi^+\pi^-$ combination in the ρ^0 meson region.
- Fig. 4.2 : a) $m_{\pi^+\pi^-}$ effective mass spectrum.
b) $m_{\pi^-\pi^-}$ effective mass spectrum.
- Fig. 4.3 : a) $m_{p\pi^-}$ effective mass spectrum.
b) $m_{p\pi^+}$ effective mass spectrum.
- Fig. 4.4 : Slope of the differential cross-section for the reaction $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$ for different $m_{3\pi}$ bins at 25 and 40 GeV/c.
- Fig. 4.5 : Dependence of the cross-section for the reaction $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$ for different 3π mass bins versus incident momentum. The data at 25 and 40 GeV/c are from this experiment, the other data are from bubble chamber experiments.
- Fig. 4.6 : Exponent n of the energy dependence of $\sigma(\pi^-p \rightarrow \pi^-\pi^-\pi^+p)$ versus $m_{3\pi}$ and of $\sigma(\pi^-p \rightarrow \pi^-p)$ from Reference 4.4).

Table 4.1

Cross-sections for the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$
versus 3π effective mass at 25 and 40 GeV/c

$m_{3\pi}$	25 GeV/c			$\sigma [\mu\text{b}]$	40 GeV/c	
	$\sigma [\mu\text{b}]$	Stat. error	Syst. error		Stat. error	Syst. error
0.8-1.0	36.6	3.3	3.7	37.3	2.2	3.7
1.0-1.1	51.6	3.6	5.2	50.9	2.3	5.2
1.1-1.2	68.4	3.4	7.0	64.5	2.2	6.5
1.2-1.3	71.7	3.1	7.6	63.5	2.1	6.4
1.3-1.4	56.5	2.6	7.0	53.9	2.1	5.5
1.4-1.5	35.6	2.3	5.0	30.6	1.5	3.1
1.5-1.6	35.3	2.1	5.2	33.2	1.5	3.4
1.6-1.8	80.9	3.8	11.0	72.5	2.1	7.4
1.8-2.0	48.5	3.2	7.0	44.0	2.1	4.6

Table 4.2

Exponent n of the energy dependence of the $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$
cross-section between 11 and 40 GeV/c ($\sigma \propto p_{\text{inc}}^{+n}$)

$m_{3\pi}$	n	$\chi^2/8$ points
0.3-1.0	-0.31 ± 0.11	7.0
1.0-1.2	-0.29 ± 0.10	10.1
1.2-1.4	-0.29 ± 0.10	10.5
1.4-1.6	-0.42 ± 0.11	21.1
1.6-1.8	-0.50 ± 0.11	12.0

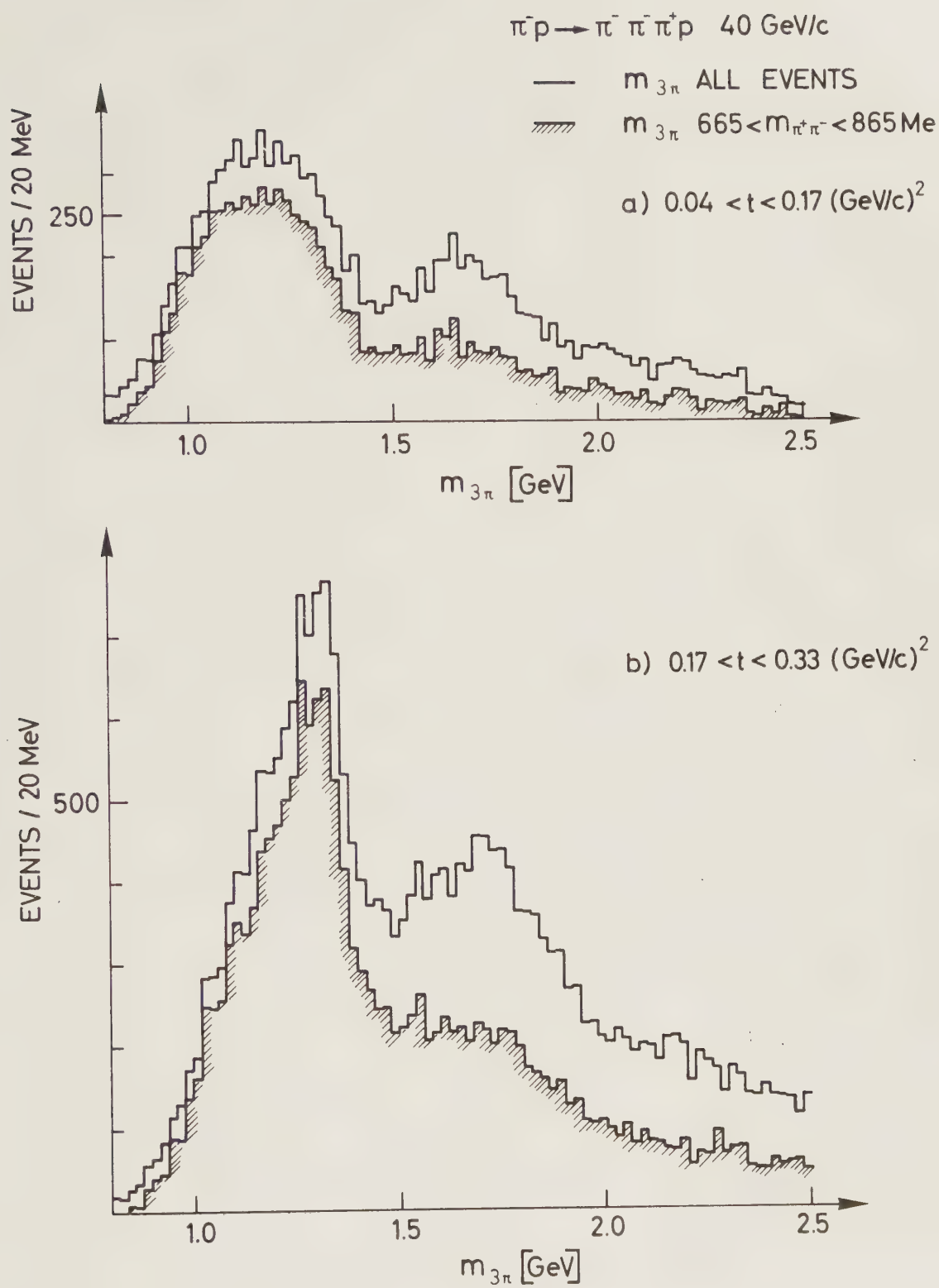


Fig. 4.1

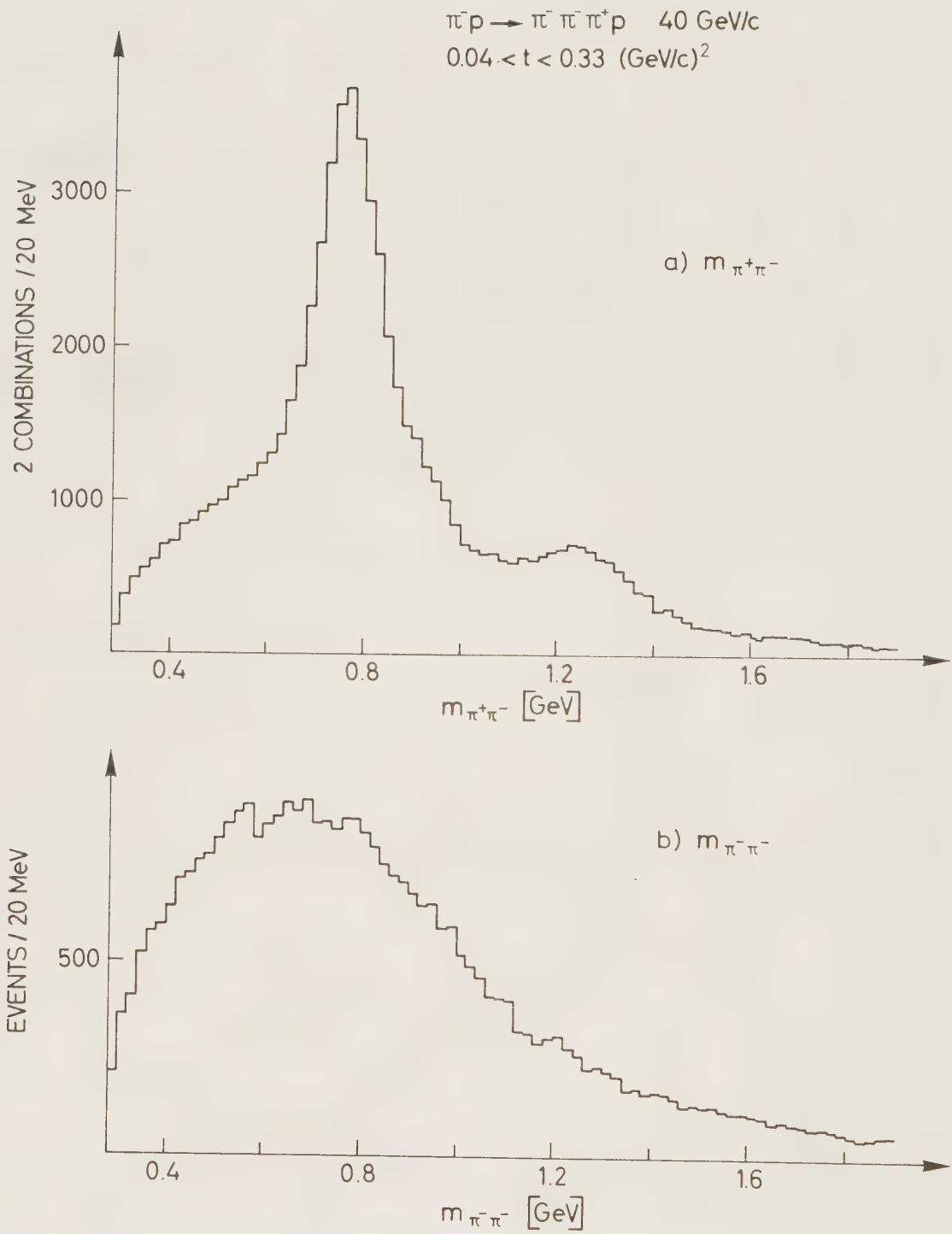


Fig. 4.2

$\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ 40 GeV/c
 $0.04 < t < 0.33 \text{ (GeV/c)}^2$

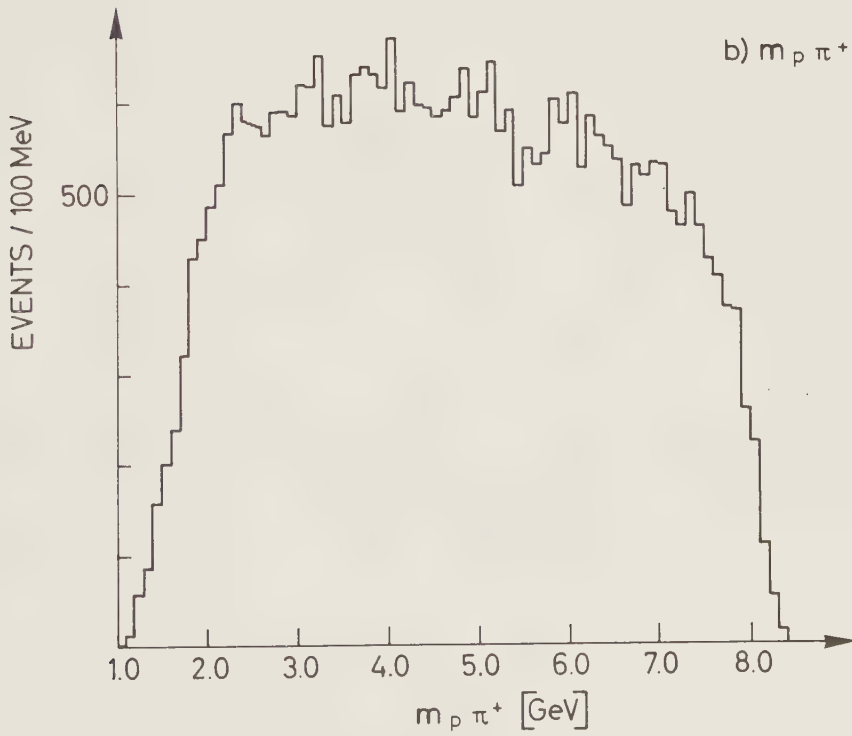
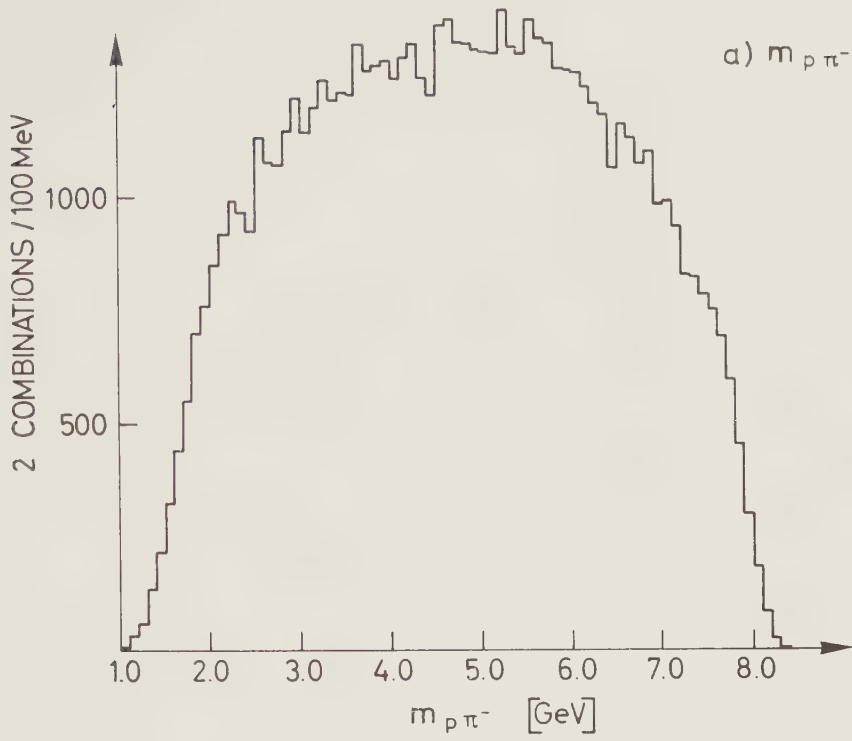


Fig. 4.3

SLOPE OF $d\sigma/dt (\pi^- p \rightarrow \pi^- \pi^- \pi^+ p)$ VERSUS $m_{3\pi}$

$(d\sigma/dt \leftarrow e^{-b|t|})$

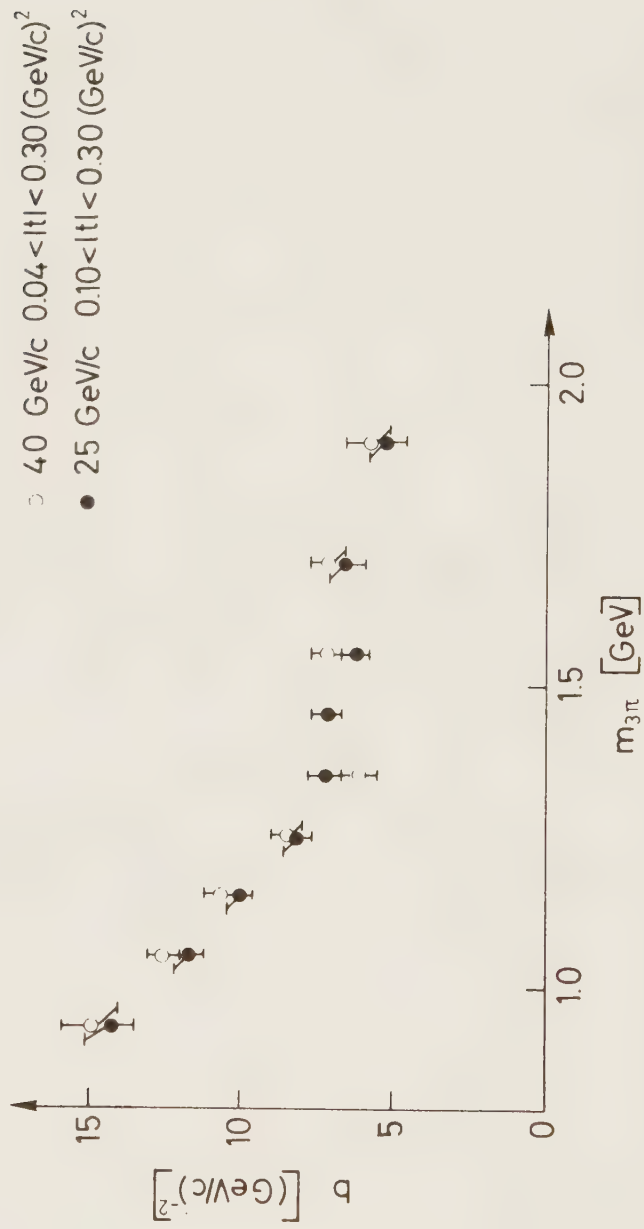


Fig. 4.4

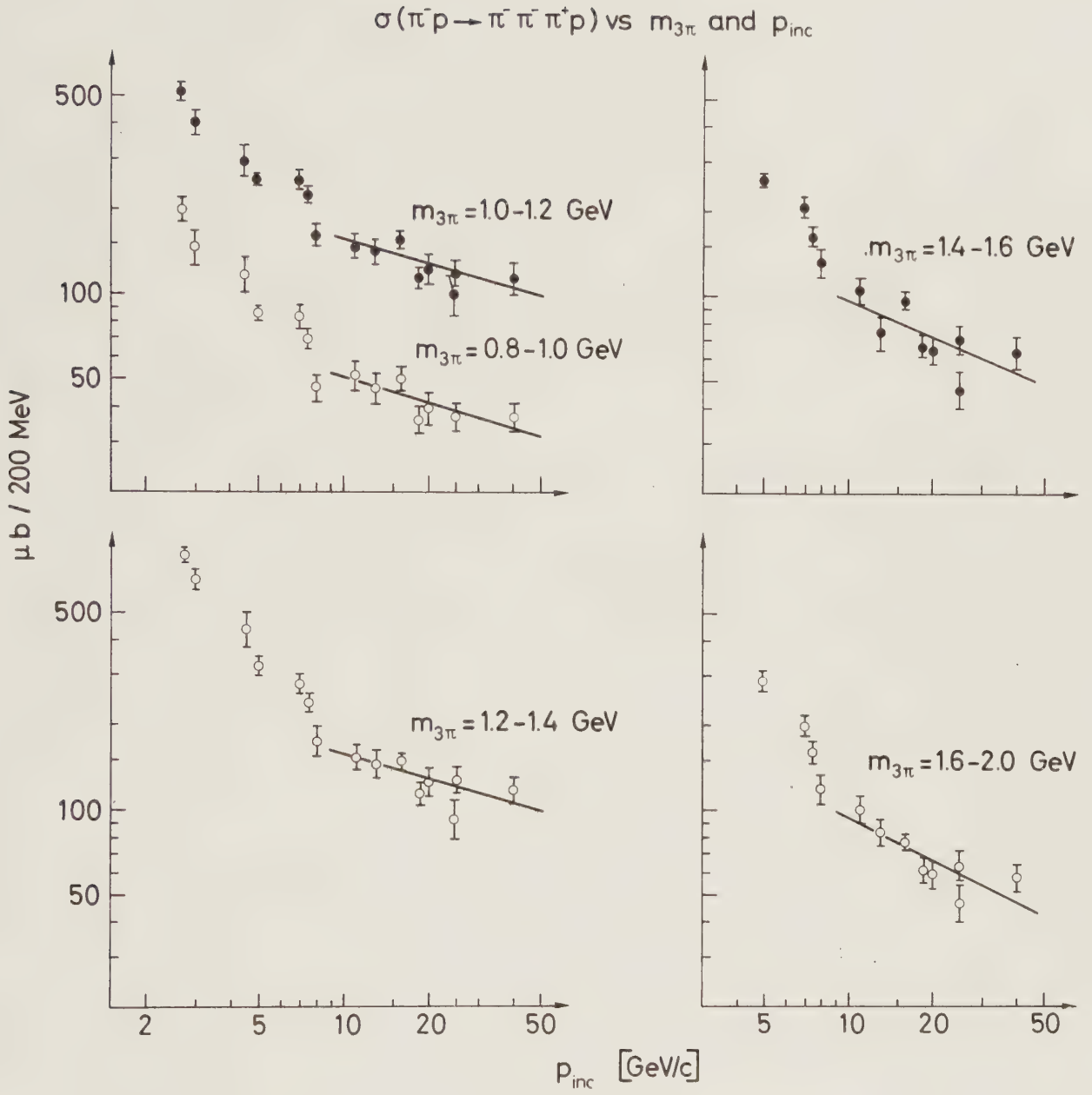


Fig. 4.5

EXPONENT n OF THE ENERGY DEPENDENCE OF
 $\sigma(\pi^- p \rightarrow \pi^- p)$ AND $\sigma(\pi^- p \rightarrow \pi^- \pi^+ \pi^- p)$ vs. $m_{3\pi}$ ($\sigma \propto p_{inc}^n$)

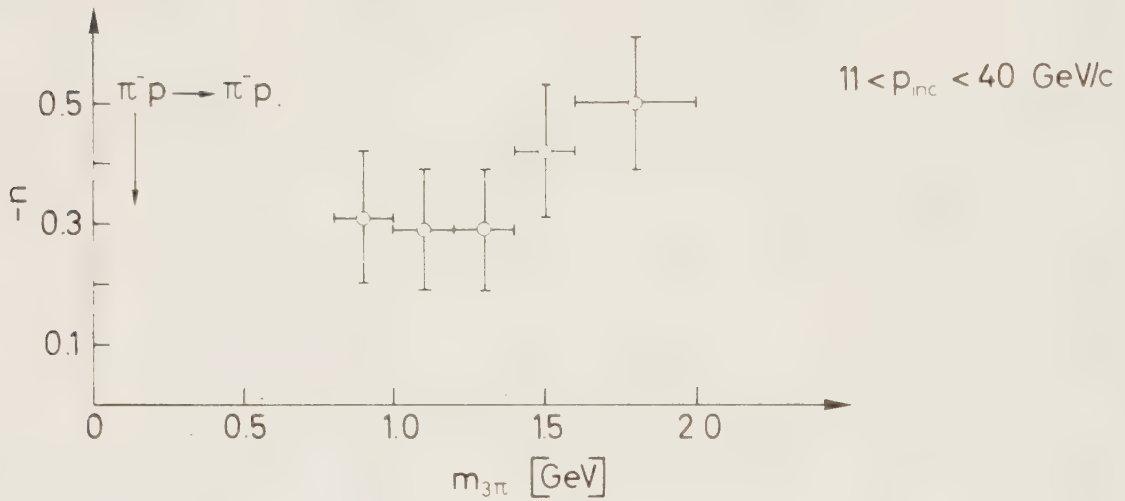


Fig. 4.6

CHAPTER 5

METHOD OF PARTIAL-WAVE ANALYSIS

In this chapter the partial-wave method^{5.1)} which we use to separate the different spin-parity states of the 3π system, their polarizations, decay modes and interferences, is described. We will first sketch the derivation of the formulae used, discuss the different assumptions made, and then briefly describe the different steps of the analysis.

5.1 QUANTUM MECHANICAL DESCRIPTION OF THE REACTION $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$

The aim of this chapter is to derive the explicit formula for the cross-section of the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$. We first note that there are nine variables that fully describe the reaction. If the incoming particle and the target particle are unpolarized, as in our experiment, the cross-section is independent of the azimuthal angle specifying the plane of the outgoing particles (i.e. $p, 3\pi$), so that we are left with eight variables only. Bearing in mind that we want to study production and decay of the 3π system, they naturally divide into two groups:

- i) Three variables to describe the production of the 3π system, for which we choose: the total centre-of-mass energy squared, s ; the momentum transfer to the proton squared, t ; the effective mass of the 3π system, $m_{3\pi}$.
- ii) Five variables to describe the decay of the 3π system, for which we choose: two Dalitz-plot variables of the 3π system, s_1, s_2 ; three Euler angles to specify the orientation of the three pions in the 3π centre-of-mass frame, $R \equiv \alpha, \beta, \gamma$.

In terms of these variables we can write the differential cross-section

$$d\sigma = W(s, t, m_{3\pi}, s_1, s_2, R) \frac{1}{2\pi} \frac{dt}{64\pi s p^{*2}} \frac{dR ds_1 ds_2}{32 m_{3\pi}^2 (2\pi)^5} dm_{3\pi}^2 \quad (5.1)$$

with p^* the centre-of-mass momentum of the incoming particles, and W the square of the Lorentz-invariant matrix element summed over outgoing and averaged over ingoing proton helicity. The phase-space terms are derived from the recursion relations for the Lorentz-invariant phase space (Lips)

$$dLips(s, P_{\pi_1}, P_{\pi_2}, P_{\pi_3}, P_p) = \frac{1}{2\pi} dLips(s, P_p, P_{3\pi}) dLips(s_{3\pi}, P_{\pi_1}, P_{\pi_2}, P_{\pi_3}) ds_{3\pi}$$

and the explicit expressions for the two- and three-body phase space

$$dLips(s, P_p, P_{3\pi}) = dt / (16\pi s^{1/2} p^*)$$

$$dLips(s_{3\pi}, P_{\pi_1}, P_{\pi_2}, P_{\pi_3}) = \frac{1}{32} \left(\frac{1}{2\pi}\right)^5 \frac{d^3R ds_1 ds_2}{s_{3\pi}}$$

[P_i denotes the four vectors; we have chosen the usual normalization for one-particle states $|P_i\rangle$ so that $\int |P_i\rangle \langle P_i| dLips(P_i) = 1.$]

We will calculate the square of the Lorentz-invariant matrix element in two steps:

- production of the 3π system,
- decay of the 3π system into three pions.

5.1.1 Production of the 3π system

In the reaction $a + b \rightarrow c + d$ [e.g. $\pi^- p \rightarrow (\pi^- \pi^- \pi^+) p$] we characterize the production of the system c by the Hermitian density matrix $\rho_c(s, t, m_c)$. We obtain ρ_c from the density matrix ρ_{ab} of the initial system and the helicity amplitudes $f_{\lambda_c \lambda_d, \lambda_a \lambda_b}$ by summing over the unobserved helicities $\lambda_a \lambda_b \lambda_d$

$$g_{\lambda_c \lambda_c'} = \sum_{\lambda_a \lambda_b \lambda_d} f_{\lambda_c \lambda_d, \lambda_a \lambda_b} g^{ab} f_{\lambda_c' \lambda_d, \lambda_a \lambda_b}^*$$

To choose a suitable complete set of basis vectors into which we develop the density matrix ρ_c , we note that the density matrix ρ_c commutes with the reflection operator $Y = \exp(-iJ_y \pi)P$ [where $\exp(-iJ_y \pi)$ is the rotation operator around the y axis, and P is the parity operator], if the y axis is chosen normal to the production plane, the incoming system is unpolarized, and parity and rotational invariance are conserved in the production process. The operator Y transforms the incoming system into a system with the same particle momenta but opposite helicities, which is as probable as the initial system, if it is unpolarized. Thus the density matrix ρ_{ab} commutes with Y . As the interaction Hamiltonian also commutes with Y , also the outgoing system commutes with Y , and summing over the helicities of particle d we find $[\rho_c, Y] = 0$, i.e. the density matrix is diagonal with respect to Y . We therefore choose, for the partial-wave decomposition, the basis vectors $|J^P M \eta\rangle$, which are eigenstates of

- $J^2(J)$, the angular momentum operator,
- $|J_z|(M)$, the third component of the $\vec{J}$ operator,
- $P(P)$, the parity operator,
- $Y(-\eta)$, the reflection operator on the production plane.

From the relation $Y|J^P M\rangle = P(-)^{J-M}|J^P -M\rangle$ we find the relation between the more familiar $|J^P M\rangle$ states and the $|J^P M \eta\rangle$ states

$$|J^P M \eta\rangle = \lambda_M \left(|J^P M\rangle + \eta P(-1)^{J+M+1} |J^P -M\rangle \right) \quad (5.2a)$$

with

$$\lambda_M = \begin{cases} \frac{1}{2} & \text{for } M=0 \\ \frac{1}{\sqrt{2}} & \text{for } M \neq 0. \end{cases}$$

From this we obtain the relation between the density matrix elements defined on the basis $|J^P M\rangle$ and $|J^P M \eta\rangle$

$$\begin{aligned} \delta_{M_1 \eta_1}^{J_1 P_1} \delta_{M_2 \eta_2}^{J_2 P_2} &= 2\lambda_{M_1} \lambda_{M_2} \left(\delta_{M_1}^{J_1 P_1} \delta_{M_2}^{J_2 P_2} + \eta_1 (-1)^{J_2 + M_2 + 1} \right. \\ &\quad \left. \cdot P_1 \delta_{M_1 - M_2}^{J_1 P_1} \delta_{-M_2}^{J_2 P_2} \right) \delta_{\eta_1 \eta_2} \end{aligned} \quad (5.2b)$$

Another reason for choosing eigenstates of Y for the basic states is that in an exchange model, in the high-energy limit $[(1/s)^2 \rightarrow 0]$, η is the naturality of the exchanged system e^{5.2)}

$$\eta = P_e (-1)^{J_e}.$$

We will see later from our data that the cross-section for the states with $\eta = (-)$ is very small compared to the states with $\eta = (+)$.

5.1.2 Decay of the 3π system

In this sub-section we will calculate the decay amplitude for the decay of a 3π system of fixed 3π mass, definite spin and parity and 3rd component of spin into 3π mesons^{5.3)}. First we discuss the choice of the five decay variables:

- i) The two Dalitz plot variables, $s_1 = m(\pi_3^+ \pi_2^-)^2$ and $s_2 = m(\pi_3^+ \pi_1^-)^2$, completely specify the energy and orientation of the three pions about their centre-of-mass.
- ii) The three Euler angles, ϕ , θ , γ , determine the orientation of the three pions with respect to fixed space axis.

For the space axis we use the Jackson axis in the 3π rest frame defined by:

$$\begin{aligned}\hat{z} &= \hat{\pi}_{\text{in}} && \text{(direction of the beam in } 3\pi \text{ rest frame),} \\ \hat{y} &= \hat{p}_{\text{out}} \times \hat{\pi}_{\text{in}} && \text{(normal to production plane),} \\ \hat{x} &= \hat{y} \times \hat{z}.\end{aligned}$$

The choice of the system is arbitrary. However, we will find experimentally, that the description of the production of the 3π system at high energies is simpler in the Jackson system than in the 3π helicity system, where the z axis is defined opposite to the direction of the recoil proton in the 3π centre-of-mass system. For the Euler angles we use the Euler angles of the odd pion (π^+) defined as (see Fig. 5.1):

- ϕ , θ , polar and azimuthal angles of the π^+ direction,
- γ , azimuthal angle of the plane of the two negative pions about the direction of the π^+ measured from the direction normal to the plane of the z -axis and the π^+ direction.

To calculate the decay distribution of the π^+ for a state $|J^P_M\rangle$, we decompose it into helicity states of the $(\pi_1^-\pi_2^-)$ system:

$$(3\pi)^- \rightarrow \pi_3^+ + (\pi_1^-\pi_2^-) \xrightarrow{\quad} \pi_1^- + \pi_2^- .$$

This is general as long as we sum over all possible helicities ν of the $(\pi_1^-\pi_2^-)$ system. We write the decay amplitude:

$$M_n^{J^P} = \langle s, s_2, \phi\theta\gamma | U | J^P_M \rangle = \langle s, s_2, 000 | R^+ U | J^P_M \rangle$$

where U is a rotationally invariant, parity conserving transition operator, and R the rotation operator. Expanding in terms of helicity states $|J^P_\nu\rangle$ of the $(\pi_1^-\pi_2^-)$ system and using

$$D_{M\nu}^{J^*}(\phi\theta\gamma) = \langle J^P_\nu | R^+ | J^P_M \rangle$$

$$g_\nu^{J^P}(s, s_2) = \langle s, s_2, 000 | U | J^P_\nu \rangle$$

we obtain

$$M_n^{J^P}(s, s_2, \phi\theta\gamma) = \sum_\nu g_\nu^{J^P}(s, s_2) \sqrt{\frac{2J+1}{8\pi^2}} D_{M\nu}^{J^*}(\phi\theta\gamma) \quad (5.3)$$

The functions $D_{M\nu}^J(\phi\theta\gamma)$ describe the angular distribution of the positive pion and follow just from conservation of angular momentum. The internal orientation of the three pions is described by the function $G_{LV}^{JP}(s_1s_2)$. It depends on the dynamics of the break-up of the 3π system. To calculate it we assume that the state $|J^P_M\rangle$ decays into a $(\pi_2^-\pi_3^+)$ subsystem with spin S , which decays subsequently into two pions. We thus get:

$$M_n^{J^P} = \sum_\lambda \langle \vec{p}_2 \vec{p}_3 | U_2 | \vec{p}_{23} \lambda \rangle \langle \vec{p}_1 \vec{p}_{23} \lambda | U_1 | J^P_M \rangle$$

where λ is the helicity of the $(\pi_2^-\pi_3^+)$ system, and U_1 and U_2 the transition operators for the two decay processes. Expanding in helicity states of well-defined angular momentum we find

$$\begin{aligned} \langle \vec{p}_1, \vec{p}_{23}, \lambda | U_1 | J^P M \rangle &= \sqrt{\frac{2J+1}{4\pi}} D_{M\lambda}^{*J}(\phi, \theta, 0) T_{S\lambda}^J \\ \langle \vec{p}_2, \vec{p}_3 | U_2 | \vec{p}_{23}, \lambda \rangle &= \sqrt{\frac{2S+1}{4\pi}} D_{\lambda 0}^{*S}(\psi, \chi, 0) T_S \end{aligned}$$

with

$$T_{S\lambda}^J(m_{3\pi}, s_1) = \langle p_1 p_{23} J M \lambda | U_1 | J^P M \rangle$$

$$T_S(s_1) = \langle p_2 p_3 | U_2 | p_{23} S \lambda \rangle.$$

Here ϕ_1 and θ_1 are the azimuthal and polar angles of the direction of the momentum of the $(\pi_2^-\pi_3^+)$ system in the 3π centre-of-mass system; ψ_1 and χ_1 are the azimuthal and polar angles of the π_3^+ in the $(\pi_2^-\pi_3^+)$ rest frame measured from the $(\pi_2^-\pi_3^+)$ -helicity axis (see Fig. 5.2). Joining these formulae we get the result:

$$M_M^{J^P} = \sum_{\lambda} \sqrt{\frac{2J+1}{4\pi}} \sqrt{\frac{2S+1}{4\pi}} D_{M\lambda}^{*J}(\phi, \theta, 0) D_{\lambda 0}^{*S}(\psi, \chi, 0) T_{S\lambda}^J T^S. \quad (5.4)$$

As we have seen from Eq. (5.3), the Dalitz-plot dependence of the decay amplitude is independent of the orientation of the three pions. We thus can choose the special orientation with $\phi = \theta = \gamma = 0$, where $\phi_1 = \pi$, $\psi_1 = \pi$, and $\theta_1 = \pi - \theta_{13}$; θ_{13} is the angle between the π_1^- and π_3^+ in the 3π rest system. Equation (5.2) reads for this special case:

$$\sum_L G_{L\nu_L}^{J^P}(s_1, s_2) = \sqrt{\frac{8J_1^2}{2J+1}} M_M^{J^P}(s_1, s_2, \phi=0, \gamma=0, \gamma=0).$$

From Eq. (5.4) we then get:

$$g_v^{JP}(s_1 s_2) = \sum_{\lambda} \sqrt{\frac{2S+1}{2}} (-1)^{J+\lambda} d_{v-\lambda}^J(\theta_{13}) d_{\lambda 0}^S(\chi_1) T_{S\lambda}^J T^S. \quad (5.5)$$

Next we will express the Dalitz-plot amplitudes, which we have calculated in terms of $(\pi_2^- \pi_3^+)$ helicity amplitudes in terms of states of defined angular momentum ℓ between the π_1^- and the $(\pi_2^- \pi_3^+)$ system. We therefore define

$$T_{S\ell}^{JP} = \langle p_1 p_2 s_1 s_2 | J M \ell S | U | J^P M \rangle \quad (5.6a)$$

and obtain

$$T_{S\lambda}^{JP} = \sum_{\ell} \sqrt{\frac{2\ell+1}{2J+1}} \langle J\lambda | \ell S 0 \lambda \rangle T_{S\ell}^{JP} \quad (5.6b)$$

$\langle J\lambda | \ell S 0 \lambda \rangle$ denotes the Clebsch-Gordan coefficient for the decomposition of a state $|J\lambda\rangle$ into states $|\ell 0\rangle$ and $|S\lambda\rangle$. This formula implies that the different decay modes of a particular 3π state are coherent. This is correct if they are decay modes of a single system like a resonance, but is wrong if they are decay modes of two incoherently produced systems with the same spin and parity.

Since the two negative pions are indistinguishable bosons, the total amplitude must be symmetric under interchange of π_1^- and π_2^- . This is achieved by adding to the Dalitz-plot amplitude calculated above the similar amplitude for the process

$$(3\pi)^- \longrightarrow (\pi_1^- \pi_3^+) + \pi_2^-$$

If we want γ to refer in both cases to the direction of the π_1^- we obtain a factor $\exp(i\pi\gamma)$ from the D function in Eq. (5.3), and obtain for the Bose-symmetric Dalitz-plot amplitude

$$\tilde{g}_v^{JP}(s_1, s_2) = g_v^{JP}(s_1, s_2) + (-1)^v g_v^{JP}(s_2, s_1) \quad (5.7)$$

So far all we have done is exact, but we now have to make simplifying assumptions to get rid of the infinite number of parameters. When applying our model to describe the measured data we will try to justify from the data some of these assumptions:

- i) Only states with a spin value of $J < J_{\max}$ are contributing to the reaction.
- ii) As it is known from $\pi\pi$ scattering^{5,4)} that the phase shift of iso-spin 2 scattering is close to zero degree ($0 > \delta_{I=2} > -30^\circ$), we neglect any interaction between the two negative pions.
- iii) Below a 2π effective mass of 1.4 GeV ($\pi^+\pi^-$) scattering can be described by s-, p-, and d-waves. We therefore restrict the analysis to these waves of the $(\pi^+\pi^-)$ system.
- iv) The still unknown functions $T_S T_{S\ell}^{JP}$ of Eqs. (5.3) and (5.5) could be determined from the data, in principle. Due to limited statistics however, we parametrize s-, p-, and d-waves of the $(\pi^+\pi^-)$ system by Breit-Wigner resonances with appropriate angular momentum barrier factors. We thus write

$$T_S T_{S\ell}^{JP} = BW^S(s_1) p^\ell q^S C_{S\ell}^{JP} \quad (5.8)$$

with

$$BW^S(s_1) = (m_s^2 - s_1 - i m_s \Gamma_s(s_1))^{-1}.$$

[Breit-Wigner amplitude for the $(\pi_3^+\pi_2^-)$ system with spin S]

$$\Gamma_s(s_1) = \gamma_s \left(\frac{q}{q_0} \right)^{2S+1} \cdot \frac{m_s}{\sqrt{s_1}}$$

(dynamic width of Breit-Wigner),

where

m_S is the mass of the $(\pi^+\pi^-)$ resonance with spin S ,

γ_S is the width of the resonance with spin S ,

q is the momentum of the decay pions in the centre of mass of the two pion resonance,

q_0 is the value of q for $s_1 = m_S^2$,

p is the momentum of the dipion resonance in the 3π centre-of-mass,

$C_{s\ell}^{JP}$ is the complex coupling constant for the decay $(3\pi^-)^- \rightarrow (\pi^+\pi^-) + \pi^-$ which depends only on the production variables $s, t, m_{3\pi}$.

With the help of Eq. (5.2) we obtain the decay amplitude for the decay of a state $|J^P M \eta\rangle$ from the amplitudes

$$m_{M\eta}^{JP} = \lambda_M \left\{ m_M^{JP} + \eta P (-1)^{J+M+1} m_{-M}^{JP} \right\} \quad (5.9)$$

and finally for the square of the Lorentz-invariant matrix element

$$W(st m_{3\pi} s_1 s_2) = \sum_{J^P M \eta} m_{M_1 \eta_1}^{J_1 P_1} \cdot g_{M_1 \eta_1}^{J_1 P_1} \cdot g_{M_2 \eta_2}^{J_2 P_2} \cdot m_{M_2 \eta_2}^{* J_2 P_2} \quad (5.10)$$

defining

$$m_{M\eta}^{JP} = \sum_{s\ell} C_{s\ell}^{JP} m_{M\eta s\ell}^{JP}.$$

Inserting the expression for W in Eq. (5.1), we obtain the differential cross-section for the reaction.

The aim of the next section is to describe how we determine from the measured data the unknown parameters as a function of the production variables of the 3π system.

5.2 FIT OF THE THEORETICAL DISTRIBUTION TO THE DATA

Up to now we have only considered the theoretical distributions for a fixed set of production variables $s, t, m_{3\pi}$. Due to finite statistics, however, we have to select for any fit a sufficiently large $m_{3\pi}$ and t bin [typically 50 MeV and 0.1 (GeV/c)^2] to have sufficient events (> 500), to

determine the unknown parameters. We take a possible variation of the parameters into account by averaging the different terms $\mathcal{M}_i$ of the decay amplitudes weighted by the experimentally determined (or guessed) $m_{3\pi}$ and t dependence $f_i(m_{3\pi}, t)$. To perform the fit to the data we make the following calculations:

- i) From the integration of the decay functions $|\mathcal{M}_i|^2$, over the phase space of the decay variables, we obtain the normalization factors $N_i(m_{3\pi})$ for different $m_{3\pi}$ values in the selected $m_{3\pi}$ bin.
- ii) Monte Carlo integration of the decay functions $\mathcal{M}_i \cdot \mathcal{M}_j^*$, weighted by their $m_{3\pi}$ and t dependence and the acceptance of the experiment, yields the acceptance integrals a_{ij} [see Chapter 3 (Section 3.3)]. They determine how much the different decay distributions are attenuated by the apparatus.
- iii) For every measured event ($k = 1, K$) with the measured variables $m_{3\pi}^k$, t^k , $\vec{t}^k$ (3π mass, momentum transfer to the proton, decay variables of the 3π system) we calculate

$$\tilde{m}_i(k) = \mathcal{M}_i(\vec{t}^k) \cdot f_i(m_{3\pi}^k, t^k) \cdot N_i.$$

In these quantities the logarithm of the likelihood reads

$$\Lambda = \sum_k \ln \sum_{ij} \tilde{C}_{ij} \tilde{m}_i(k) \tilde{m}_j(k) - K \sum_{ij} \tilde{C}_{ij} a_{ij}$$

(with $\tilde{C}_{ij} = C\rho C^*$, the indices $J^P S \ell$ for the C 's and $J^P M \eta$ for the density matrix elements ρ).

To obtain the best fit we maximize Λ with respect to the unknown parameters ρ and C . As the probability distribution has to be positive definite at all times, we cannot vary the density matrix elements arbitrarily but we have to require that ρ is a positive definite matrix. Whenever during an iteration the maximum of Λ is found outside the positivity domain of the density matrix, we change to the system where the density matrix is diagonal and prevent its eigenvalues from becoming negative.

Having thus found the maximum of Λ we calculate the errors on the obtained parameters by approximating Λ by a second-order polynomial, around its maximum, and assume that Λ changes by half a unit, if one parameter is changed by one standard deviation.

To determine how well our model describes the data we generate Monte Carlo events according to the best estimates of the parameters, weight them according to the acceptance of the apparatus, and compare different one- and two-dimensional distributions with the data.

REFERENCES

- 5.1) The program to perform the partial-wave analysis was developed at the University of Illinois by G. Ascoli, D.V. Brockway, L. Eisenstein, J.D. Hansen, M.L. Ioffredo, U.E. Kruse and P.F. Schultz. In the following publications the authors applied the method to data of the reaction $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$.
G. Ascoli et al., Phys. Rev. Letters 26, 929 (1971).
D.V. Brockway, Study of the three pion final state interactions in the reaction $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$ at 5 and 7.5 GeV/c, University of Illinois report COO-1195-197 (1970).
G. Ascoli et al., Summary of the experimental situation regarding A1, A2 and A3 production, submitted to the 16th Int. Conf. on High-Energy Physics, Batavia 6-13 September 1972.
- 5.2) W. Wittek, Untersuchung der Reaktionen $K^+n \rightarrow K^+\pi^-p$ bei 3 GeV/c $K^+n \rightarrow K^{*0}p$ bei 2.3 GeV/c und 3 GeV/c und $K^-p \rightarrow K^{*0}n$ im Primär-impulsbereich von 2 bis 11 GeV/c, Dissertation, Universität München (1971).
- 5.3) Suh Urk Chung, Spin formalism, CERN 71-8 (1971).
H. Pilkuhn, The interactions of hadrons (North-Holland, Amsterdam, 1972).
- 5.4) J.L. Basdevant, C.D. Frogatt and J.L. Petersen, $\pi\pi$ -phenomenology below 1100 MeV, preprint CERN-TH 1519.

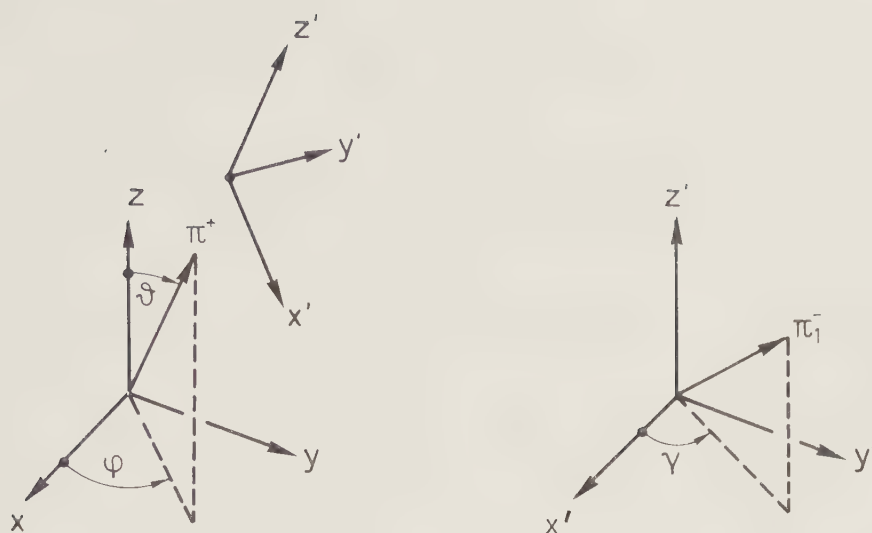
FIGURE CAPTIONS

Fig. 5.1 : Definition of the Euler angles of the π^+ in the decay $(3\pi)^- \rightarrow \pi^- + \pi^- + \pi^+$; ϕ and θ are the azimuthal and polar angles of the π^+ in the 3π centre-of-mass system; γ is the orientation of the 3π plane in the helicity system of the π^+ .

Fig. 5.2 : Definition of the decay angles for the decay

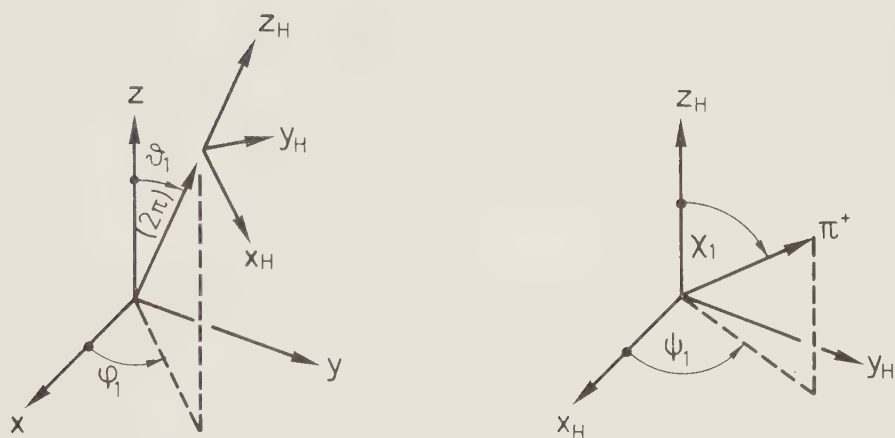
$$(3\pi)^- \rightarrow \pi_1^- + (\pi_2^+ \pi_3^-) \rightarrow \pi_2^+ + \pi_3^-$$

The azimuthal and polar angles of the $(\pi_2^+ \pi_3^-)$ system are denoted by ϕ_1 and θ_1 . The azimuthal and polar angles of the π^+ from the decay of the $(\pi_2^+ \pi_3^-)$ system in the $(\pi_2^+ \pi_3^-)$ helicity system are denoted by ψ_1 and χ_1 .



DEFINITION OF THE EULER ANGLES (φ & γ) OF THE π^+

Fig. 5.1



DEFINITION OF THE DECAY ANGLES FOR THE DECAY

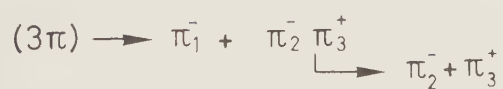


Fig. 5.2

CHAPTER 6

RESULTS OF THE PARTIAL-WAVE ANALYSIS

BELOW A 3π MASS OF 1.5 GeV

In this chapter we will use the model, developed in the last chapter, to describe the data of the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ at 25 GeV/c and 40 GeV/c as a function of momentum transfer to the proton and $\rho\pi$ effective mass. The aim is to determine the unknown parameters of our model from the data, and to discuss their physical meaning:

- i) The density matrix $\rho_{M_1 \eta_1}^{J_1 P_1} \rho_{M_2 \eta_2}^{J_2 P_2}$ [see formula (5.10)], the real diagonal elements of which measure the probability of producing the 3π system in a state with the quantum numbers $J^P M \eta$ and the complex off-diagonal elements of which determine the amount and phase of interference between the two states with the quantum-numbers $J_i^{P_i} M_i \eta_i$, where $i = 1, 2$;
- ii) The complex ratios $C_{S_1 \ell_1}^{J^P} / C_{S_2 \ell_2}^{J^P}$, which measure the phase and relative strength of the decay of a 3π system with the quantum numbers J^P into $(\pi^+ \pi^-)$ systems with the spin S_i and the angular momentum ℓ_i between the third pion and the $(\pi^+ \pi^-)$ system.

6.1 PARTIAL-WAVE ANALYSIS OF THE DATA

In Table 6.1 we list the allowed ℓ -wave decays of the 3π system for $S \leq 2$, and $J \leq 3$. We find 17 complex ratios of coupling constants, and thus 34 real parameters. For every J^P state there are $2J + 1$ polarizations possible, so that the density matrix for $J \leq 3$ is a 31×31 Hermitian matrix. Taking into account the constraints of conservation of parity and the normalization condition (the trace of the density matrix is equal to 1), we find 480 real elements, and thus a total of 514 real parameters. Next we will show that actually most of these parameters are zero within the statistical sensitivity of our data, and that the number of non-vanishing parameters is sufficiently small to enable the performance of the analysis.

6.1.1 Determination of the maximum spin and the maximum angular momentum

To find out the maximum spin value of the 3π system we perform a Fourier analysis of several angular distributions for different 3π mass bins^{*)}. Below a 3π mass of 1.4 GeV, the coefficients of the terms $\cos(nx)$ and $\sin(nx)$ for $n > 4$ are statistically not significant. Above 1.4 GeV the terms with $n = 6$ and $n = 8$ start to become detectable. We thus conclude that, in the mass interval considered, the maximum spin is 2, and we therefore fix the intensity of the states with $J > 2$, $S > 2$, and $\ell > 2$ to zero, and are left with 159 free parameters.

6.1.2 Choice of the basic hypothesis

The choice of the basic hypothesis was strongly influenced by the results of a bubble-chamber experiment performed at lower beam energies^{6.1)}. We will try to make the choice plausible by discussing some angular distributions from our 40 GeV/c data. For the data in the 3π mass interval from 1.0 to 1.2 GeV and 1.2 to 1.4 GeV, we show the distribution of the polar angle θ of the π^+ from the $\rho^0 \rightarrow \pi^+\pi^-$ decay in Fig. 6.1. The angle is measured in the 3π rest frame from the direction of the incident π^- meson. Apart from the strong asymmetry, the distribution follows about $P_1^2(\cos \theta)$ ^{**)} , which we expect for the s-wave decay of a state with the polarization $M = 0$ into $\rho\pi$. We thus conclude the presence of a strongly produced $J^P = 1^+$, $M = 0$ state. We interpret the strong asymmetry of the distribution (the minimum is around 0.2), as the interference between the decays of the $\rho \rightarrow \pi^+\pi^-$ and $\varepsilon \rightarrow \pi^+\pi^-$, known from $\pi\pi$ scattering. We therefore expect both the s-wave ($\rho\pi$) and p-wave ($\varepsilon\pi$) decay mode of the $J^P = 1^+$ state to be present which we denote by $1^+(S/P)0^+$. The last + is the eigenvalue η of the reflection operator on the production plane, which can assume only the value +1 in this case [see formula (5.2)]. Inspection of the flat azimuthal distribution of the π^+ in the 3π rest frame, measured from the Gottfried-Jackson axes (Fig. 6.2) suggests that states with the polarization

*) We used the 40 GeV/c data, for which the acceptance function is sufficiently smooth, so that the analysis can be made directly on the uncorrected data.

**) $P_1(x) = x$ is the first Legendre polynomial.

$M \neq 0$ are unimportant. We further include in our hypothesis the state $0^-(S/P)0^+$ (i.e. a state with $J^P = 0^+$ decaying via s- and p-wave into $\epsilon\pi$ and $\rho\pi$). We do not include a state with $J^P = 1^-$, as we know, from Ref. 6.1, that it is only weakly produced at lower beam momenta. We will later determine how big the contribution of the 1^- -state is. For the 2^- -state we assume that it decays via p- and s-waves into $\rho\pi$ and $f\pi$, and that its polarization is $M = 0$. We further assume that below a 3π mass of 1.5 GeV, which is relatively close to $\rho\pi$, $\epsilon\pi$ and $f\pi$ thresholds, d-, f-, and higher waves are suppressed by angular momentum barrier factors. From the effective mass spectra (Fig. 4.1), we have seen indication of A2 production, which is the 2^+ state decaying via d-wave into $\rho\pi$. From the analysis of A2 production in the reaction $\pi^- p \rightarrow K^- K_S^0 p$ at incident momentum around 20 GeV/c^{6.2)} we know that the A2 is produced at high energies in a pure $M = 1$ and $\eta = +1$ -state. We thus include also the partial-wave $2^+(D)1^+$. Summarizing, our basic hypothesis reads, in the notation $J^P(\ell)M\eta$:

$$\begin{aligned} &0^-(S/P)0^+ \\ &1^+(S/P)0^+ \\ &2^+(D)1^+ \\ &2^-(S/P)0^+. \end{aligned}$$

The unknown parameters are the 16 real density matrix elements, and the three complex ratios of coupling constants, i.e. there are a total of 22 real parameters. Before we actually determine how well this basic hypothesis describes the data, and that it is the unique solution of the partial-wave decomposition, we will first check the assumptions made about the $(\pi^+\pi^-)$ system.

6.1.3 Parametrization of the $(\pi^+\pi^-)$ system

To investigate the dependence of the quality of the likelihood-fit and the sensitivity of our results to the parametrization of $\pi^+\pi^-$ scattering [see formula (5.8)], we fit the data with different parametrizations and compare the values of the likelihood, and check the stability of the solution.

p-wave of $(\pi^+\pi^-)$ system

The p-wave of the $(\pi^+\pi^-)$ system is known to be dominated by the ρ resonance. From the Particle Data tables^{6.3)} we find for its mass and width

$$m_\rho = 765 \pm 10 \text{ MeV} ; \Gamma_\rho = 125 \pm 20 \text{ MeV}.$$

The relatively big errors are due to differences in production mechanisms in the different reactions, method of analysis, and uncertainties in the parametrization of the resonance shape. We use a relativistic p-wave Breit-Wigner amplitude

$$A_p(s_1) = \frac{1}{(m_\rho^2 - s_1 - i m_\rho \gamma_\rho)}$$

$$\gamma_\rho(s_1) = \Gamma_\rho \left(\frac{q}{q_0}\right)^3 \frac{m_\rho}{\sqrt{s_1}}$$

where

s_1 is the $\pi^+\pi^-$ effective mass squared,

q is the momentum of the decay pions in the $(\pi^+\pi^-)$ centre of mass,

q_0 is the value of q for $s_1 = m_\rho^2$,

Γ_ρ is the width of the ρ meson.

In Fig. 6.3a we show the maximum of the likelihood function we obtain when fitting our basic hypothesis to the data at 25 GeV/c in the 3π mass interval 1.0-1.2 GeV, and the momentum transfer interval 0.10-0.33 (GeV/c)² for different values of m_ρ and Γ_ρ . The maximum value of the logarithm of the likelihood $\mathcal{L}_{\max}$ is obtained for the parameters

$$m_\rho = 769 \text{ MeV}, \Gamma_\rho = 138 \text{ MeV}.$$

Also shown in the figure are the curves $\Delta\mathcal{L} = \mathcal{L}_{\max} - \mathcal{L} = 2$, which correspond to two standard deviations (σ) from the best estimate, and the curve $\Delta\mathcal{L} = 4$ (i.e. $\sim 3\sigma$). From this, we estimate the statistical error

$$\Delta m_\rho = \pm 3 \text{ MeV}, \Delta \Gamma_\rho = \pm 5 \text{ MeV}.$$

Within ± 5 MeV, the same values for m_ρ and Γ_ρ are found from the data in the mass interval 1.2-1.4 GeV and the data at 40 GeV/c. The influence of the exact values of m_ρ and Γ_ρ on the 3π parameters is small: they change by less than one standard deviation when changing m_ρ and Γ_ρ by 10 MeV around their best estimate.

s-wave of $(\pi^+ \pi^-)$ system

It is more difficult to find a good parametrization of the s-wave of the $(\pi^+ \pi^-)$ system, as there is no unique answer from $\pi^+ \pi^-$ scattering measured in π -exchange reactions. We therefore used two different parametrizations for the s-wave scattering amplitude

$$A_s = (\eta_s e^{2i\delta_s} - 1) / 2i$$

where δ_s is the s-wave phase shift and η_s is the s-wave inelasticity.

ϵ resonance

$$\begin{aligned} A_s(s_1) &= 1 / (m_\epsilon - s_1 - im_\epsilon \gamma_\epsilon) \\ \gamma_\epsilon(s_1) &= \Gamma_\epsilon \left(\frac{q}{q_0} \right) \cdot \frac{m_\epsilon}{\sqrt{s_1}} \\ \eta_s(s_1) &= 1 \end{aligned}$$

with the same symbols as used for the p-wave parametrization, described above. Figure 6.3b shows the distribution of the maximum of the logarithm of the likelihood obtained from fitting the data [25 GeV/c, $m_{3\pi} = 1.0-1.2$ GeV, $t = 0.10-0.33$ (GeV/c)²] with different values of m_ϵ and Γ_ϵ . We obtain the best fit for the values

$$m_\epsilon = 760 \pm 25 \text{ MeV}; \quad \Gamma_\epsilon = 370 \pm 30 \text{ MeV}.$$

Curve I of Fig. 6.4 shows the dependence of the phase shift δ_s on the $\pi^+ \pi^-$ effective mass for this solution. It agrees roughly with the "up" solution from $\pi^+ \pi^-$ scattering experiments^{6.4}).

"Down" solution of the s-wave of $(\pi^+\pi^-)$ system

In a recent paper Protopopescu et al.^{6.4)} claim that a simultaneous analysis of the reactions $\pi^+\pi^- \rightarrow \pi^+\pi^-$ and $\pi^+\pi^- \rightarrow K^+K^-$ enables them to find a unique solution for the $\pi\pi$ s-wave phase shift below a dipion mass of 1.15 GeV. We therefore also tried to fit our data with their solution, which is displayed as curve II in Fig. 6.4. In this parametrization we also allow for an inelasticity η_s smaller than 1, for dipion masses above the K^+K^- threshold. We fitted all our data in the 3π mass intervals 1.0-1.2 GeV and 1.2-1.4 GeV. In both cases the fit is somewhat worse (2 to 3σ) than the fit obtained for the parametrization with an s-wave resonance. There is no significant change in the results obtained.

d-wave of $(\pi^+\pi^-)$ system

We describe the d-wave of $\pi^+\pi^-$ scattering, which becomes important for $\pi^+\pi^-$ masses above 1.1 GeV, by the f meson. For mass and width we take the values from the Particle Data tables:

$$m_f = 1265 \text{ MeV}; \Gamma_f = 154 \text{ MeV}.$$

6.1.4 Quality of the maximum likelihood fit

The aim of this sub-section is to show that the solution found by the fit, together with the choice of the basic hypothesis, is the unique solution of the partial-wave decomposition of the 3π data with the model discussed in Chapter 5.

To demonstrate that the fit is a good description of the data, we generate Monte Carlo events according to the best estimates of the parameters found from the likelihood fit, weight them by the acceptance and efficiencies of the apparatus, and compare them with the measured data. As we know of no general method to determine the statistical compatibility of two distributions in many dimensions we compare several one-dimensional projections. In Figs. 6.1 and 6.2, four histograms of the data at 40 GeV/c in the 3π mass interval 1.0 to 1.2 GeV and 1.2-1.4 GeV are compared with the corresponding Monte Carlo distributions (smooth curve). From the good agreement we conclude that our model describes the data well.

To check that our result is the unique solution of the partial-wave analysis for the chosen hypothesis, we performed several fits, starting from different initial values of the unknown parameters. Whenever the fit converged, it converged to the same solution.

To find out whether the choice of our hypothesis (i.e. fixing 490 out of 512 parameters to zero) is unique we used the following observation. From the angular distribution of Fig. 6.1 it is clear that the low-mass 3π system is dominated by the $1^+(S)$ -state. This agrees with the results from other experiments at lower beam energies^{6.1)}, and the fact that we cannot get an acceptable fit by fixing the intensity of the $1^+(S)$ -state at zero but allowing most of the other states of Table 6.1 to vary freely. From the rank condition on the density matrix, which we will discuss in Section 6.2, we know that most of the different spin-parity states are bound to interfere. This interference is the source of the strong asymmetry observed in the angular distribution of Fig. 6.1. We thus conclude that the presence of partial waves, which are produced with small cross-section compared to the main waves, will show up via their interference with the dominant waves, and not via their own intensities. We therefore claim that we can study their production cross-section and their influence on the other partial waves by adding them separately to our basic hypothesis. This we did for all states and polarizations mentioned in Table 6.1 for the two mass bins 1.0-1.2 GeV and 1.2-1.4 GeV at 25 and 40 GeV/c. Table 6.2 summarizes the results. The contributions of all states (apart from the $M = 1$ polarization of the 1^+ -state) omitted in our basic hypothesis are small, and change only slightly (by < 1 standard deviation) the parameters of the other partial waves. We therefore believe that we have found the unique solution for the partial-wave analysis of the 3π system for our model.

Next we will discuss the results of the partial-wave analysis. We will first quote a few general results before treating in detail the properties found for the 1^+ -state (A1 meson) and 2^+ -state (A2 meson).

6.2 GENERAL RESULTS OF THE PARTIAL-WAVE ANALYSIS

6.2.1 Mass dependence of the different partial waves

From Table 6.2 we see that the only significant partial wave we omitted in the "basic hypothesis" is the $1^+(S/P)1^+$ -state, which we now add to our hypothesis

$$0^-(S/P)0^+, 1^+(S/P)(0^+/1^+), 2^+(D)1^+, 2^-(S/P)0^+.$$

The number of free parameters of this hypothesis is 31. To determine their mass dependence, we fit this hypothesis to the data at 25 and 40 GeV/c in 50 MeV bins in the 3π mass interval between 0.9 and 1.5 GeV. Figures 6.5a and 6.5b show the intensity of the different partial waves. The results at both energies are very similar. Below 1.4 GeV the 3π system is dominated by the A1 system, the 1^+ -state decaying via s-wave into a ρ meson and a π meson. It peaks around 1.2 GeV, and has a full width of about 300 MeV. The 2^+ -state decaying via d-wave into $\rho\pi$ (A2 meson), peaks slightly above 1.3 GeV. The 2^- -state, which decays via s-wave into $f\pi$ (A3 system), starts to rise around 1.5 GeV. All other partial waves show no significant structures. The intensities of the states not drawn (and not varied in the above fit) are shown in Table 6.2 for the two 3π mass intervals 1.0-1.2 and 1.2-1.4 GeV. As a result, following empirical rules for the production of 3π spin-parity states at high energies in the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$, are found

- i) partial waves produced via unnatural parity exchange (i.e. $\eta = -1$) are strongly suppressed,
- ii) with the exception of the 2^+ -state, all strongly produced states have the same naturality $[P(-)^J]$ as the incident π meson (i.e. $J^P = 0^-, 1^+, 2^-, \dots$),
- iii) only the states with the smallest polarization in the Gottfried-Jackson system (allowed by natural parity exchange) are strongly produced^{*)}.

*) According to formula (5.2a), the $M = 0$ polarization of the J^P series $1^-, 2^+, 3^-, \dots$ can be produced only by unnatural parity exchange at high energies.

6.2.2 Rank of the over-all density matrix

In this sub-section we will show by simple arguments that the rank of the density matrix of the 3π system cannot exceed four in the general case, and two if unnatural parity exchange can be neglected.

The rank of the 3π density matrix is equal to the number of independent transition amplitudes which can produce the final state in the reaction. As the proton has spin $\frac{1}{2}$, there are exactly four possibilities in our case

- | | | |
|-------------------------------|---|---|
| - helicity flip of the proton | } | natural parity exchange ^{*)} , |
| - helicity non-flip | | |
| - helicity flip | } | unnatural parity exchange. |
| - helicity non-flip | | |

Experimentally we have found that both unnatural parity exchange amplitudes are negligible at 25 and 40 GeV/c, and we therefore expect the rank of the 3π density matrix to be smaller or equal to two (this is only exact if we do not average over other variables than the helicity of the proton, such as 3π effective mass or momentum transfer to the proton). Experimentally, we measure the rank of the density matrix by the number of non-vanishing eigenvalues of the density matrix. In Fig. 6.6 the three biggest eigenvalues out of the five of our hypothesis of the 3π density matrix calculated from the fit of Sub-section 6.2 are shown versus 3π effective mass. We see that the condition $\text{rank} \leq 2$ is well satisfied; we also note that, independent of the structures in the intensities of the single partial waves, there is one eigenvalue which dominates, the value of which decreases smoothly from 0.95 to 0.7 in the 3π mass interval between 1.0 and 1.6 GeV. We thus conclude that there is one dominant production amplitude with respect to the proton spin variables.

*) Here unnatural (natural) parity exchange is only a brief way of saying that the production amplitude does not change (changes) sign under reflection on the production plane. The above argument is also valid at finite energies, where unnatural and natural parity exchange have no exact meaning.

6.3 RESULTS CONCERNING THE PRODUCTION AND DECAY OF THE A1 MESON

6.3.1 Experimental situation

The present experimental situation regarding the A1 meson is unclear: all experiments which measure the reactions $\pi^\pm p \rightarrow \pi^\pm \pi^- \pi^+ p$ at small momentum transfer to the proton find a significant enhancement in the 3π effective mass spectrum with masses between 1.05 and 1.15 GeV, and a width around 0.3 GeV, very similar to our data shown in Fig. 4.1. This peak can be explained qualitatively by models of the diffractive production of 3π mesons^{6.5,6.6}). The partial wave analysis of Ascoli et al.^{6.1}) finds 1^+ for the spin parity of the diffractive peak, with subsequent s-wave decay into $\rho\pi$. Some experiments claim, in addition to the diffractive peak, that there is a narrow peak ($m \simeq 1070$ MeV, $\Gamma \simeq 50$ MeV), which however is statistically never very significant. The search for this narrow structure was therefore extended to reactions in which the diffractive background is absent as in backward production, proton-antiproton annihilation, and charge-exchange reactions. Some experiments have found indications for a narrow structure with the correct parameters but with low statistical significance.

We conclude that there is no convincing evidence of a narrow A1 meson, but only for a wide, diffractively-produced peak with the spin parity 1^+ , which dominates the low mass 3π systems in the reactions $\pi^\pm p \rightarrow \pi^\pm \pi^- \pi^+ p$ at low momentum transfers.

6.3.2 Spin, parity, isospin, G-parity, mass, width and differential cross-section of the A1 enhancement

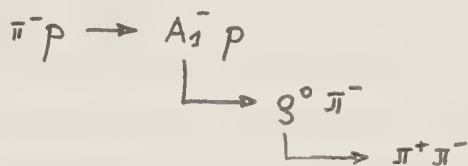
From the 3π effective mass spectra of Fig. 4.1 we find no indication for a narrow structure around 1070 MeV in our experiment (the average mass resolution is 12 MeV at this 3π effective mass). We therefore restrict our discussion to the diffractively-produced enhancement, which we call the A1 meson.

From Fig. 6.5 we see that the 1^+ -state, which decays via s-wave into $\rho^0 \pi^-$, is the only state which peaks in the region around 1.2 GeV. Thus the spin parity of the A1 is 1^+ and its decay mode $\rho\pi$. As we observe the negatively-charged 3π spectrum, and there is no experiment which has

seen the A1 in a doubly-charged channel, we conclude that the isospin is 1, and its G-parity negative.

We noticed in Chapter 4 (Section 4.2) that the slope b of the differential cross-section depends strongly on the 3π effective mass in the A1 region. In Fig. 6.7 we show the differential cross-section for production of the $1^+(S)$ -state for the two mass bins 1.0 to 1.2 GeV and 1.2 to 1.4 GeV. We find again this strong dependence for the $1^+(S)$ -state separated from the other states (see table below). As a result, the value of mass and width of the A1 system depends on the momentum transfer interval selected. Until now no quantitative explanation has been found for this phenomenon, and it is therefore impossible to determine uniquely mass and width of the A1 system.

Assuming that the function $\exp(b * t)$, which describes the differential cross-section in the accepted t interval is valid for all values of t , we calculate the total cross-section for the reaction



	25 GeV/c		40 GeV/c	
$\Delta m_{3\pi}$ [GeV]	b [(GeV/c) ⁻²]	$\sigma(A1)$ [μb]	b [(GeV/c) ⁻²]	$\sigma(A1)$ [μb]
1.0-1.2	-12.1 ± 1.1	92 ± 10 (± 10)	-11.9 ± 1.1	72 ± 5 (± 7)
1.2-1.4	-8.0 ± 1.0	59 ± 4 (± 6)	-6.8 ± 0.8	52 ± 3 (± 6)

The systematic error in parenthesis, which is larger than the statistical error, contains the uncertainties of the cross-section normalization, and of the determination of the ratio A1 events/total number of events.

6.3.3 Polarization of the A1 meson

According to formula (5.2b) we can write the polarization density matrix for the 1^+ -state in the notation $\rho_{M_1 M_2 \eta}$

$$\begin{pmatrix} g_{00+} & g_{10+} & 0 \\ g_{10+}^* & g_{11+} & 0 \\ 0 & 0 & g_{11-} \end{pmatrix}.$$

From our data at 40 GeV/c in the mass interval $\Delta m_{3\pi} = 1.0-1.2$ GeV, we obtain the following values for the parameters in the Gottfried-Jackson system

ρ_{00+}	0.95 ± 0.01
ρ_{11+}	0.05 ± 0.01
$\text{Re } \rho_{01+}$	-0.18 ± 0.01
$\text{Im } \rho_{01+}$	0.00 ± 0.04

We note that the A1 is mainly produced in an $M = 0$ state, the $M = 1$ component being small but statistically significant. To discuss the physical meaning of these parameters, and their dependence on 3π mass and momentum transfer t to the proton, we rotate the density matrix to the system where it is diagonal^{6,7)} (this system is usually called the Donohue-Høgaassen system). The complex rotation angle ψ and the eigenvalues λ are obtained from the formulae:

$$\begin{aligned} \tan(2\psi) &= -2g_{10+} / (g_{00+} - g_{11+}) \\ \lambda_{1,2+} &= \frac{g_{00+} + g_{11+}}{2} \left(1 \pm \sqrt{1 + 4 \frac{|g_{10+}|^2 - g_{00+}g_{11+}}{g_{00+} + g_{11+}}} \right) \\ \lambda_- &= g_{11-} \end{aligned}$$

and for the parameters we obtain from the measured density matrix elements

$$\begin{aligned} \lambda_{1+} &= 0.985 \pm 0.01 \\ \lambda_{2+} &= 0.015 \pm 0.01 \\ \lambda_- &= 0.00 \pm 0.01 \\ \psi &= \text{Re}(\psi) = 11.3 \pm 0.7^\circ \\ &\quad \text{Im}(\psi) = 0.0 \pm 3.0^\circ. \end{aligned}$$

We see that in the system, which is reached by a rotation of θ around the normal of the production plane from the Gottfried-Jackson system, the A1 is within the experimental errors in a pure $M = 0$ state, and we can discuss the polarization density matrix with the help of the single parameter θ . In Fig. 6.8 we show the dependence of θ on the momentum transfer for the 3π mass intervals 1.0-1.2 GeV and 1.2-1.4 GeV for the 40 GeV/c data. The rotation angle θ_c between the direction of the beam particle and the direction opposite to the direction of the recoil proton in 3π centre of mass (crossing angle) is indicated as a solid line. In the kinematic region investigated the angle θ is roughly constant, and has a value of about 10° . Similar results have been found at 25 GeV/c.

6.3.4 Interference of the A1 system with other partial waves, and resonance interpretation

The aim of this sub-section is to demonstrate that we can find out whether a definite partial wave resonates, and to show that the $1^+(S)$ -state does not resonate around 1.15 GeV.

In Section 6.2.2 we have seen that we observe strong interference between the different partial waves. The absolute value of the off-diagonal density matrix element $\rho^{J_1^{P_1} J_2^{P_2}}$ measures the amount of interference between the states $J_1^{P_1}$ and $J_2^{P_2}$, its argument ϕ the phase difference. The phase of a partial-wave amplitude is the product of its production phase times its decay phase. Assuming that the production phases are slowly varying functions of the 3π mass, ϕ measures, apart from a constant, the difference between the two decay phases. If the $J_1^{P_1}$ system is a resonance with the parameters m_1 and Γ_1 interfering with a non-resonant wave $J_2^{P_2}$, we expect an increase of ϕ by 90° when moving in 3π mass from $m_1 - (\Gamma_1/2)$ to $m_1 + (\Gamma_1/2)$. We will show in Section 6.4 that we actually find this behaviour for the A2 meson. The dominant background waves $0^-(S)$, $1^+(P)$ and $2^-(P)$ do not peak below the A1 peak; we thus suggest that they are not resonating and provide constant phases to measure the variation of the decay phase of the A1 system. In Fig. 6.9 we show the interference phases. They are all constant at the peak of the $1^+(S)$ -state, and we conclude that the A1 object cannot be described by an ordinary Breit-Wigner amplitude.

6.4 RESULTS CONCERNING THE PRODUCTION AND DECAY OF THE A2 MESON

6.4.1 Experimental situation

The A2 meson is one of the best-established boson resonances. Its mass is known to be 1310 ± 10 MeV, its width 100 ± 20 MeV ^{6.3)}. The controversy about a possible structure is not yet completely settled, although most of the high-statistics experiments favour a Breit-Wigner shape. The resolution of this experiment is too bad, and its statistics too low to contribute anything to this problem. The main decay modes of the A2 meson are ^{6.3)}

$$\begin{aligned} 3\pi & \dots (76.8 \pm 1.8)\% \\ \eta\pi & \dots (16.3 \pm 1.5)\% \\ K\bar{K} & \dots (5.8 \pm 0.8)\% . \end{aligned}$$

In the reactions $\pi p \rightarrow \eta\pi p$ and $\pi p \rightarrow K\bar{K}p$ the A2 signal appears almost without background, and the analysis of these channels is relatively simple; in the reaction $\pi p \rightarrow \pi\pi\pi p$ it appears on a background which is 1.5 to 3 times bigger than the signal and the analysis is more complicated, but we can obtain more information about its production and decay. The main interest at present in studying the A2 meson is to determine its production mechanism at high energies. It is now well-established that the differential cross-section decreases in the forward direction^{6.1,6.2)}, both from analysis of the $K\bar{K}$ decay mode and the $\rho\pi$ decay mode. All experiments also show that the A2 is produced at high energies by natural parity exchange with a pure $M = 1$ polarization in the Gottfried-Jackson system. There is, however, a controversy about the momentum dependence of its production cross-section. Whereas the data from the reaction $\pi^- p \rightarrow K^- \bar{K}^0 p$ ^{6.2)} yield a dependence $p_{\text{inc}}^{-1.3}$, the partial-wave analysis^{6.1)} of the 3π decay mode yields $p_{\text{inc}}^{-0.7}$.

6.4.2 Spin, parity, isospin, G-parity, mass, width, and production cross-section of the A2 meson

From Fig. 6.5 we find that the 2^+ -state, which decays via d-wave into 3π , is the only partial wave peaking around 1.3 GeV. Thus we find 2^+ for the spin and parity of the A2. As we observe the single-charged

3π system, and as there is no experiment which has found a double-charged A_2 , we conclude that its isospin is 1, and its G-parity negative.

In Fig. 6.10 we show the dependence of the 2^+ intensity on the 3π effective mass for the 40 GeV data at high momentum transfer. Fitting a relativistic d-wave Breit-Wigner function (BW) to the data we obtain

$$BW(m) = m \cdot m_{A_2} \cdot \Gamma(m) / \{ (m^2 - m_{A_2}^2)^2 + m_{A_2}^2 \Gamma^2(m) \}$$

$$\Gamma(m) = \Gamma_{A_2} (q/q_0)^{2\ell+1} \quad (\ell=2)$$

where q is the centre-of-mass momentum of the decay products of the A_2

$$m_{A_2} = 1315 \pm 5 \text{ MeV}; \quad \Gamma_{A_2} = 115 \pm 15 \text{ MeV}.$$

The values of the parameters agree well with the results from other experiments^{6.3}). Because of the uncertainty of the resonance parametrization far from the resonance mass, we confine further discussion to the events ± 100 MeV around the peak of the A_2 ($1200 < m_{3\pi} < 1400$ MeV). In Fig. 6.11 we show the differential cross-section for the production of the $2^+(D)$ -state. There is a significant drop of the differential cross-section at low momentum transfers. The function $d\sigma/dt = \sigma b^2 t e^{-bt}$ describes the data well. From our data we obtain for the parameters σ and b :

	σ [μb] *)	b [(GeV/c) $^{-2}$]
25 GeV/c	15 ± 3 (± 2)	8.5 **)
40 GeV/c	18 ± 2 (± 2)	8.6 ± 1.2

*) The quantity in parenthesis is the estimated systematic error.

**) As the momentum transfer interval is too small at 25 GeV/c to determine b , we fixed it at 8.5 (GeV/c)^{-2} .

Integration over the full momentum transfer range yields the total cross-section for A_2 meson production, under the assumption that the above formula is valid also outside the measured t -range.

6.4.3 Polarization of the A2 meson

For the elements of the density matrix of the A2 meson we find

	ρ_{11+}	ρ_{22+}	Re ρ_{21+}	Im ρ_{21+}	Other elements
25 GeV/c	0.90 ± 0.04	0.06 ± 0.05	0.12 ± 0.04	0.01 ± 0.12	< 0.05
40 GeV/c	0.93 ± 0.04	0.07 ± 0.04	0.10 ± 0.03	-0.03 ± 0.08	< 0.05

The only element which is significantly different from zero is ρ_{11+} . We thus conclude that the A2 meson is produced by natural parity exchange with the polarization $M = 1$ at high energies. This agrees with the results found around 20 GeV/c in the reaction $\pi^- p \rightarrow K^- K_S^0 p$ (6.2).

6.4.4 Interference with other partial waves and resonance interpretation of the A2 system

Next we want to test whether the A2 is a resonance or not, by investigating the interference of the $2^+(D)$ -wave with other partial waves. We show different interference phases in Fig. 6.10. They all increase by roughly 90° , when the 3π effective mass changes from $m_{A2} - \Gamma_{A2}/2$ (1.26 GeV) to $m_{A2} + \Gamma_{A2}/2$ (1.36 GeV). The solid curve displays the expected variation of the interference phases, assuming constant production and decay phases for the non-resonant partial waves, and constant production phase and a Breit-Wigner decay amplitude for the 2^+ -state. We thus conclude that the A2 is a resonance which can be well described by a Breit-Wigner amplitude.

6.5 DISCUSSION OF THE RESULTS

In this section we compare the results of this experiment with the findings of experiments performed at lower beam energies, and discuss the data in the framework of a simple Regge-pole parametrization.

6.5.1 Properties of A1 and A2 systems

The results of this experiment are very similar to the results of the partial-wave analysis of bubble-chamber data^{6.1)} in the energy interval between 5 and 25 GeV.

- i) The $J^P = 2^+$ state resonates at a mass of 1315 ± 5 MeV, and has a full width of 115 ± 15 MeV. This state is usually called the A2 meson.
- ii) The intensity of the $1^+(S)$ -state, usually called the A1 meson, has a resonance like shape with the approximate parameters $m \simeq 1150$ MeV and $\Gamma \simeq 300$ MeV. The exact parameters depend on the momentum transfer interval selected. From the measured interference phase of the $1^+(S)$ -state with the background states we can exclude that the $1^+(S)$ -state resonates around 1150 MeV. The Deck model^{6.5,6.6}, which describes the production of a low-mass 3π system by a virtual dissociation of the incident π meson into a dipion system and a third π meson, and a subsequent elastic-like scattering of the pion (or dipion system) on the proton predicts an s-wave $\rho\pi$ enhancement around 1150 MeV, which does not resonate. The parameters of the model can be adjusted so that it describes well the intensity of the $1^+(S)$ -state. It fails, however, in its simple form to predict correctly the intensities of the other partial waves, the phase angles between the different states, and the rapid variation of the slope parameter of the differential cross-section for low 3π masses.

6.5.2 Properties of the production of A1 and A2 in the reaction $\pi^-p \rightarrow \pi^-\pi^-\pi^+p$

The differential cross-section for reactions of the type $a + b \rightarrow a' + b'$, where a and b are the incoming particles, and a' and b' are the outgoing particles or particle systems, is frequently parametrized as

$$\frac{d\sigma}{dt}(s,t) = F(t) \cdot \left(\frac{s}{s_0}\right)^{2\alpha(t)-2}.$$

This formula is derived from the model of exchange of a single Regge pole, in which case $F(t)$ is an unknown function containing the Regge-pole residuum, the signature factor and possible kinematic factors, $\alpha(t)$ is the Regge trajectory function, $s_0 = 1 \text{ GeV}^2$ a dimension factor. Experimentally it has been found that in reactions, where one Regge pole is exchanged, the trajectory functions are approximately linear

$$\alpha(t) = \alpha_0 + \alpha' t$$

the value of the Regge slope α' being about 1 (GeV/c)^{-2} , the intercept α_0 smaller or equal to 0.5. The Regge-parametrization has also been extended to reactions like elastic scattering, although their production mechanism is certainly different. As most high-energy hadron collisions are inelastic, they are bound to produce shadow (or diffraction) scattering in the elastic channel. The trajectory for diffraction scattering is frequently called the Pomeron trajectory, and diffractive scattering is often called "Pomeron exchange". The parameters of the Pomeron trajectory are quite different from the ordinary Regge trajectories. Its intercept is close to one, to explain constant total cross-sections at high energies, its slope between 0 and 0.6.

Next we determine the parameters of the Regge trajectories for A1 and A2 production, from the results of the partial-wave analysis of bubble-chamber data between 11 and 25 GeV/c, and our data at 25 and 40 GeV/c.

We define the A1 as the $1^+(S)$ -state in the mass interval between 1.0 and 1.2 GeV. At each energy, the differential cross-section for A1 production was determined by the partial-wave analysis. The integrated cross-section is obtained by integrating the differential cross-section, and the slope b of the differential cross-section by fitting the expression $\exp(b * t)$ to the measured differential cross-sections. At all energies this function describes the data well. Figure 6.12 shows the dependence of the cross-section σ and the slope b on the total centre-of-mass energy squared s . Since the differential cross-section can be described as an exponential for all values of s , and $F(t)$ in formula (6.1) is the differential cross-section for $s = s_0 = 1 \text{ GeV}$, we parametrize

$$F(t) = a_0 e^{b_0 t}.$$

Using the parametrization $\alpha(t) = \alpha_0 + \alpha' t$ for the Regge trajectory and

$$\sigma = \int_{t_{\min}}^{1 \text{ (GeV/c)}^{-2}} \frac{d\sigma}{dt} dt$$

where $t_{\min}$ is the minimum momentum transfer for production of the A1 system for the total centre-of-mass energy squared s , we fit the expression (6.1) to the data of Fig. 6.12 and obtain for the best estimates of the parameters

$$\alpha_0 = 0.82 \pm 0.08$$

$$\alpha' = 0.85 \pm 0.35 \text{ (GeV/c)}^{-2}$$

$$\chi^2 = 15 \text{ for 10 degrees of freedom.}$$

We define the A2 as the $2^+(D)$ -state in the mass interval $m_{3\pi} = 1.2$ to 1.4 GeV. From Fig. 6.11 we see that the differential cross-section for A2 production decreases at small momentum transfer t . If the A2 is produced in a pure $M = 1$ state in the full momentum transfer interval, we expect its cross-section to decrease proportionally to $(t - t_{\min})$ close to the minimum momentum transfer $t_{\min}$. We thus obtain the values of the slope b of the differential cross-section by fitting the expression $(t - t_{\min}) * \exp(b * t)$ to the data. We calculate the total cross-sections σ by integrating the differential cross-section. Figure 6.13 shows the s -dependence of σ and b . We obtain the parameters of the Regge-trajectory $\alpha(t) = \alpha_0 + \alpha' t$, as well as the values of a_0 and b_0 of the expression $F(t) = a_0 * \exp(b_0 * t)$, by fitting formula (6.1) multiplied by $(t - t_{\min})$ to the data of Fig. 6.13

$$\alpha_0 = 0.90 \pm 0.12$$

$$\alpha' = 1.3 \pm 0.4 \text{ (GeV/c)}^{-2}$$

$$\chi^2 = 10.9 \text{ for 10 degrees of freedom.}$$

Within the relatively large errors α_0 and α' are similar for A1 and A2 production, and we therefore expect the same production mechanism for both reactions. This interpretation is also supported by Fig. 6.14, which shows results concerning the interference between $1^+(S)$ - and $2^+(D)$ -states in the region of the A2 meson:

i) The coherence between the two states, defined as

$$\frac{|g^{1^+S} g^{2^+D}|}{\sqrt{g^{1^+S} g^{1^+S} g^{2^+D} g^{2^+D}}}$$

is close to one, the maximum value allowed by the positivity condition of the density matrix.

- ii) The phase angle between the two amplitudes is independent of the momentum transfer t within the large experimental errors (in a simple Regge model the variation of the production phase directly measures the slope of the Regge trajectory).
- iii) The variation of the phase angle between 2_D^+ and 1_S^+ amplitudes minus the decay phase of 90° from the A2 resonance is shown in Fig. 6.14c versus incident momentum p_{inc} . Although the measurement errors are large, there is a significant decrease in phase from about -45° at 5 GeV/c to -75° at 40 GeV/c, which can be parametrized as

$$\varphi^{2^+D \ 1^+S}(p_{inc}) = (-20 \pm 12)^\circ - (16 \pm 5)^\circ \ln\left(\frac{p_{inc}}{1 \text{ GeV/c}}\right).$$

Although it is not yet clear how this phase variation can be understood, it is a strong constraint on models describing A1 and A2 production at high incident momenta.

REFERENCES

- 6.1) G. Ascoli et al., Summary of the experimental situation regarding A1, A2, and A3 production, submitted to the 16th Int. Conf. on High-Energy Physics, Batavia 6-13 September 1972.
- 6.2) K.J. Foley, W.A. Love, S. Ozaki, E.D. Platner, A.C. Saulys, E.H. Willen and S.J. Lindenbaum, Phys. Rev. D6, 747 (1972).
G. Grayer, B. Hyams, C. Jones, P. Schlein, W. Blum, H. Dietl, W. Koch, H. Lippmann, E. Lorenz, G. Lütjens, W. Männer, J. Meissburger, U. Stierlin and P. Weilhammer, Phys. Letters 34 B, 333 (1971).
- 6.3) Particle properties, Review of particle properties, Phys. Letters, April 1972.
- 6.4) S.D. Protopopescu, M. Alston-Garnjost, A. Barbaro-Galtieri, S.M. Flatté, J.H. Friedman, T.A. Lasinski, G.R. Lynch, M.S. Rabin and F.T. Solmitz, A $\pi\pi$ phase-shift analysis from reactions $\pi^+p \rightarrow \pi^+\pi^-\Delta^{++}$ and $\pi^+p \rightarrow K^+K^-\Delta^{++}$ at 7.1 GeV, *in* Proc. 3rd Philadelphia Conference on Experimental Meson Spectroscopy (1972), p. 17.
G. Grayer, B. Hyams, C. Jones, P. Schlein, W. Blum, H. Dietl, W. Koch, E. Lorenz, G. Lütjens, W. Männer, J. Meissburger, W. Ochs, U. Stierlin and P. Weilhammer, $\pi\pi$ phase-shift analysis from an experiment $\pi^-p \rightarrow \pi^-\pi^+n$ at 17.2 GeV/c, *in* Proc. 3rd Philadelphia Conference on Experimental Meson Spectroscopy (1972), p. 5.
- 6.5) R.T. Deck, Phys. Rev. Letters 13, 169 (1964).
E.L. Berger, Phys. Rev. 166, 1525 (1968).
- 6.6) M.L. Ioffredo, G.W. Brandenburg, A.E. Brenner, B. Eisenstein, L. Eisenstein, W.H. Johnsen, Jr., J.K. Kim, M.E. Law, B.M. Salzberg, J.H. Scharenguivel, L.K. Sisterson and J.J. Szymanski, Phys. Rev. Letters 21, 1212 (1968).
A.M. Cnops, P.V.C. Hough, F.R. Huson, I.R. Kenyon, J.M. Scarr, I.O. Skillicorn, H.O. Cohn, R.D. McCulloch, W.M. Bugg, G.T. Condo and M.M. Nussbaum, Phys. Rev. Letters 21, 1609 (1968).
E.L. Berger, Phys. Rev. 179, 1567 (1969).
- 6.7) J.T. Donohue and H. Høgaassen, Phys. Letters 25 B, 554 (1967).

TABLE CAPTIONS

- Table 6.1 : Allowed orbital angular momenta for the decay of a 3π system with spin parity J^P into a $(\pi\pi)$ system of spin S and the third pion. According to the known $\pi\pi$ resonances, we denote by ϵ , ρ , and f the two-pion systems with spin $S = 0, 1, 2$.
- Table 6.2 : Cross-sections (in per cent of total events) and significance of partial waves omitted in the basic hypothesis for the $m_{3\pi}$ intervals 1.0-1.2 GeV and 1.2-1.4 GeV for the data at 25 and 40 GeV/c.

FIGURE CAPTIONS

- Fig. 6.1 : Distribution of the polar angle θ of the π^+ from the $\rho^0 \rightarrow \pi^+\pi^-$ decay relative to the Gottfried-Jackson axes for the two mass intervals $m_{3\pi} = 1.0-1.2$ GeV and 1.2-1.4 GeV. The histograms are the data measured at 40 GeV/c, the lines are the results of the maximum likelihood fit.
- Fig. 6.2 : Distribution of the azimuthal angle ϕ of the π^+ relative to the Gottfried-Jackson axes for the 40 GeV/c in the two mass intervals $m_{3\pi} = 1.0-1.2$ GeV and 1.2-1.4 GeV. The histograms are the measured data, the solid lines are the results of the maximum likelihood fit.
- Fig. 6.3 : Map of the logarithm of the likelihood as a function of the parametrization of $\pi\pi$ scattering for the 25 GeV/c data:
- a) mass and width of the ρ meson;
 - b) mass and width of the ϵ meson.

- Fig. 6.4 : Parametrizations of the $\pi\pi$ s-wave:
Curve I - ϵ resonance with the parameters $m_\epsilon = 750$ MeV,
 $\Gamma_\epsilon = 400$ MeV.
Curve II - $\pi\pi$ phase-shift solution from Ref. 6.4.
- Fig. 6.5 : Intensities for the dominant partial waves versus 3π effective mass:
a) from the 25 GeV/c data;
b) from the 40 GeV/c data.
- Fig. 6.6 : Experimental determination of the rank of the density matrix of the 3π system. The rank is defined as the number of non-vanishing eigenvalues. In the graph we show the values of the three biggest eigenvalues (out of the five of the hypothesis) versus 3π effective mass.
- Fig. 6.7 : The differential cross-section for production of the $1^+(S)$ -state at 25 and 40 GeV/c for the two 3π effective mass intervals 1.0-1.2 GeV and 1.2-1.4 GeV. The curve drawn is a fit of the function e^{-bt} to the data. The values found for b are discussed in the text.
- Fig. 6.8 : Polarization of the $1^+(S)$ -state versus momentum transfer squared in the two 3π mass intervals 1.0-1.2 GeV and 1.2-1.4 GeV. The angle between the system where the $1^+(S)$ -wave is in a pure $M = 0$ state (Donohue-Høgaassen $\equiv$ D.H.) and the Gottfried-Jackson system (G.J.) is plotted.
- Fig. 6.9 : Interference phase of the $1^+(S)$ -state with other states at 25 and 40 GeV/c. If the $1^+(S)$ -state resonates at the A_1 mass (~ 1150 MeV) with a full width of ~ 300 MeV as suggested from Fig. 6.5, all phase angles should increase by 90° between 1000 and 1300 MeV.

- Fig. 6.10 : a) Mass dependence of the $2^+(D)$ -wave. The curve is the fit of a relativistic d-wave Breit-Wigner function without background to the data.
- b) Interference phase between the $2^+(D)$ -state and $1^+(S)$ - and $1^+(P)$ -wave. If the 2^+ -state resonates around 1300 MeV with a full width of 100 MeV as suggested from (a) we expect both phases to increase by 90° between 1250 and 1350 MeV. The solid curves are calculated from the Breit-Wigner fit of (a) assuming a constant phase for the $1^+(S)$ - and $1^+(P)$ -wave and adjusted vertically to fit the data.

- Fig. 6.11 : Differential cross-section for A_2 production at 25 and 40 GeV/c for the mass interval $1.2 < m_{A_2} < 1.4$ GeV. The 40 GeV/c data are fitted by the expression $d\sigma/dt = ct e^{-bt}$. The results of the fit are quoted in the text.

- Fig. 6.12 : Energy dependence of A_1 production in the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$. The A_1 is defined as the $1^+(S)$ -state in the 3π mass interval between 1.0 and 1.2 GeV:
- a) Energy dependence of the total cross-section.
- b) Energy dependence of the slope of the differential cross-section. The curve is a simple Regge fit assuming a linear Regge trajectory.

- Fig. 6.13 : Energy dependence of A_2 production in the reaction $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$. The A_2 is defined as the $2^+(D)$ -state in the 3π mass interval between 1.2 and 1.4 GeV:
- a) Energy dependence of the total cross-section.
- b) Energy dependence of the slope b of the differential cross-section. The value of b has been obtained from the data by fitting the function $(t - t_{\min}) e^{-bt}$ to the differential cross-sections. The curve is a simple Regge fit assuming a linear Regge trajectory.

Fig. 6.14 : Results on the interference of A1 and A2 in the region of the A2 resonance:

a) Coherence, defined as

$$\sqrt{\frac{g^{1^+s} 2^+D}{g^{1^+s} 1^+s} g^{2^+D} 2^+D}}$$

versus momentum transfer.

b) Phase angle, defined as

$$\arg \left(g^{2^+D} 1^+s \right)$$

versus momentum transfer.

c) Phase angle minus the decay phase of the A2 meson at the peak of the A2 resonance versus incident momentum.

Table 6.1

Allowed orbital angular momenta for the decay of a 3π system with the spin parity J^P into a $(\pi\pi)$ system of spin S and the third pion. According to the known $\pi\pi\pi$ resonances, we denote by ϵ , ρ and f the two-pion systems with spin $S = 0, 1, 2$

J^P	0^-	1^+	1^-	2^+	2^-	3^+	3^-
$\epsilon\pi$	S	P	-	-	D	F	-
$\rho\pi$	P	S/D	P	D	P/F	D	F
$f\pi$	D	P/F	D	P/F	S/D	P/F	D

Table 6.2

Cross-sections (in per cent of total events) and significance of partial waves omitted in the basic hypothesis for the $m_{3\pi}$ intervals 1.0-1.2 GeV and 1.2-1.4 GeV for the data at 25 and 40 GeV/c

State $J^P(\ell)(M)\eta$	$m_{3\pi} = 1.0-1.2 \text{ GeV}$		$m_{3\pi} = 1.2-1.4 \text{ GeV}$	
	W/T *)	S **)	W/T	S
$1^+(D)(0/1)+$	0.011 ± 0.002	8	0.008 ± 0.002	6
$1^+(S/P)(1)+$	0.040 ± 0.005	28	0.050 ± 0.006	18
$1^+(S/P)(1)-$	0.000 ± 0.004	0	0.002 ± 0.004	1
$1^-(P)(1)+$	0.010 ± 0.005	3	0.005 ± 0.010	1
$1^-(P)(0/1)-$	0.000 ± 0.010	0	0.002 ± 0.010	0
$2^-(D/F)(1/2)+$	0.010 ± 0.020	7	0.020 ± 0.020	2
$2^-(D/F)(1/2)-$	0.000 ± 0.050	0	0.010 ± 0.030	0
$2^+(P)(1/2)+$	0.002 ± 0.006	1	0.020 ± 0.008	4
$2^+(P)(0/1/2)-$	0.000 ± 0.020	0	0.010 ± 0.015	0
$3^+(F/D)(0/1/2/3)+$	0.020 ± 0.010	4	0.040 ± 0.010	7
$3^+(F/D)(1/2/3)-$	0.000 ± 0.005	0	0.000 ± 0.005	0

*) Cross-section ratio state/total.

**) Statistical significance in standard deviations (calculated from the obtained maximum of the likelihood function).

40 GeV/c $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$

POLAR ANGLE OF π^+

$665 < m_{\pi^+ \pi^-} < 865$ MeV

$1.0 < m_{3\pi} < 1.2$ GeV

$1.2 < m_{3\pi} < 1.4$ GeV

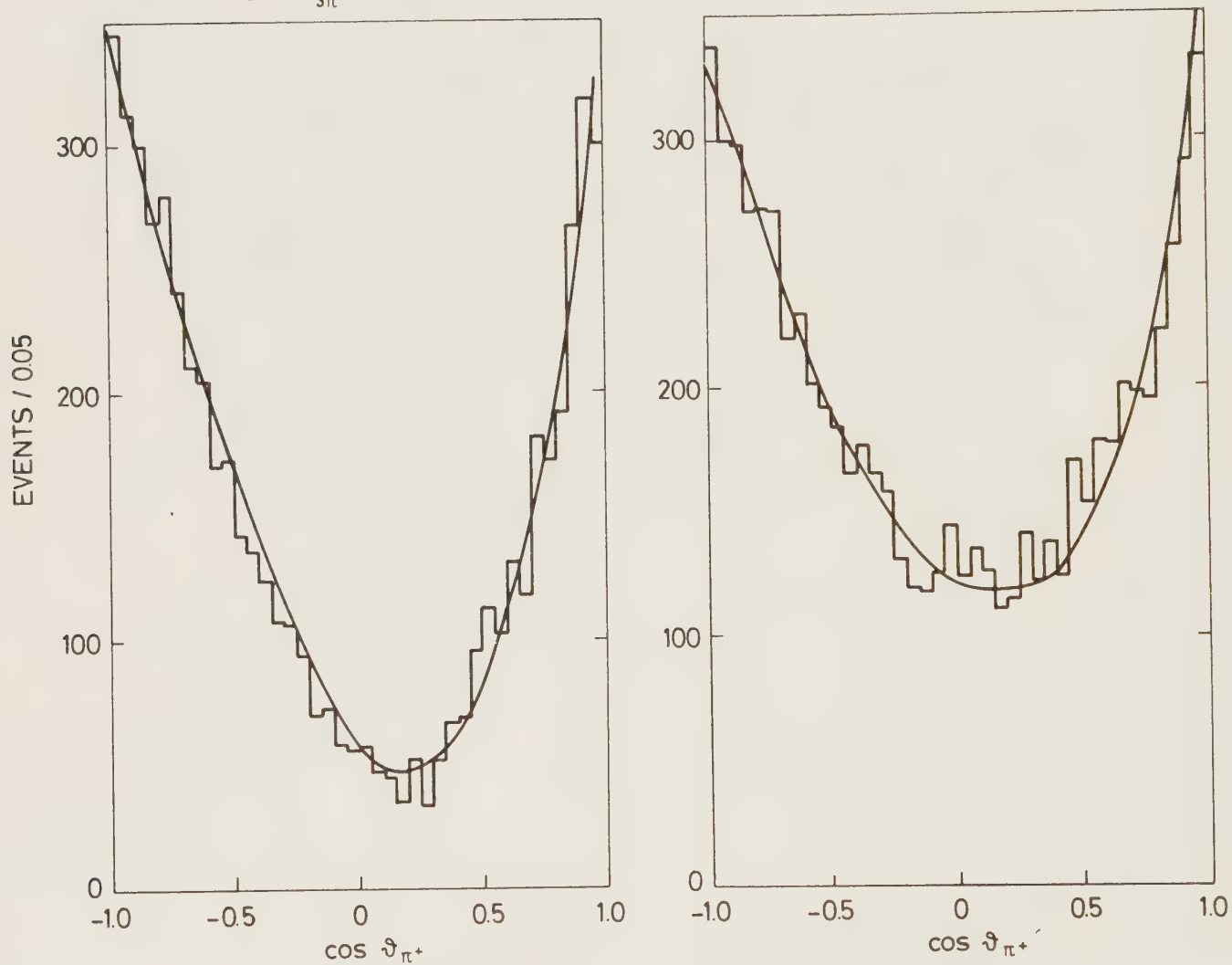


Fig. 6.1

40 GeV/c $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$
AZIMUTHAL ANGLE OF π^+

$1.0 < m_{3\pi} < 1.2$ GeV

$1.2 < m_{3\pi} < 1.4$ GeV

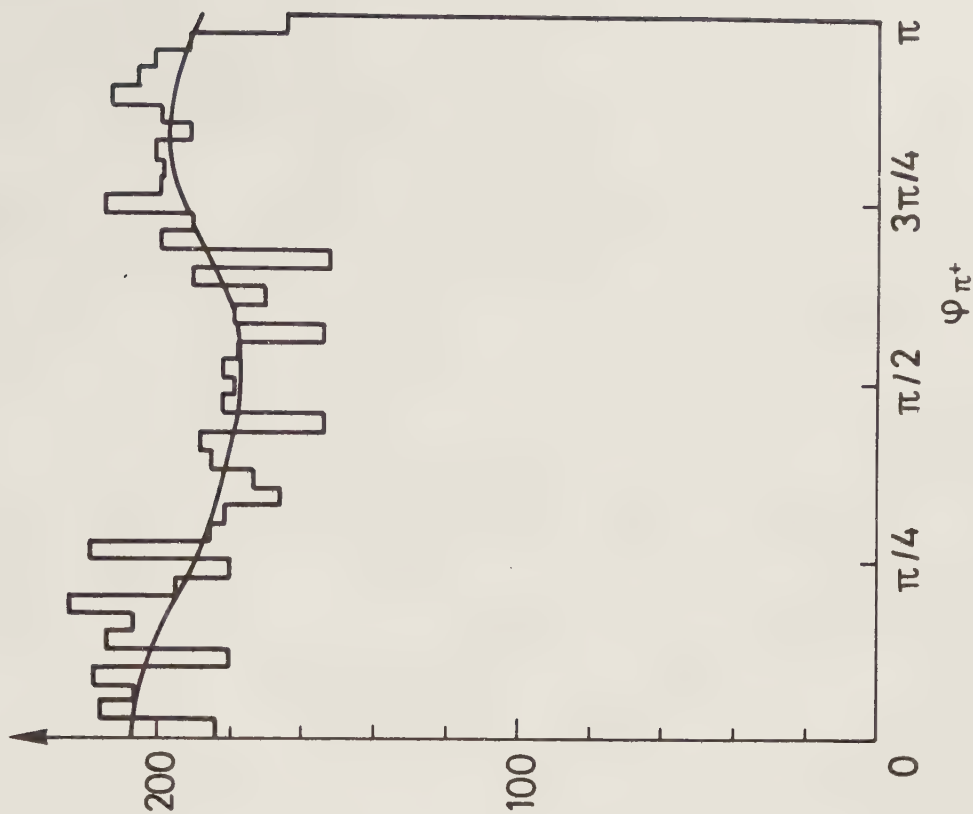
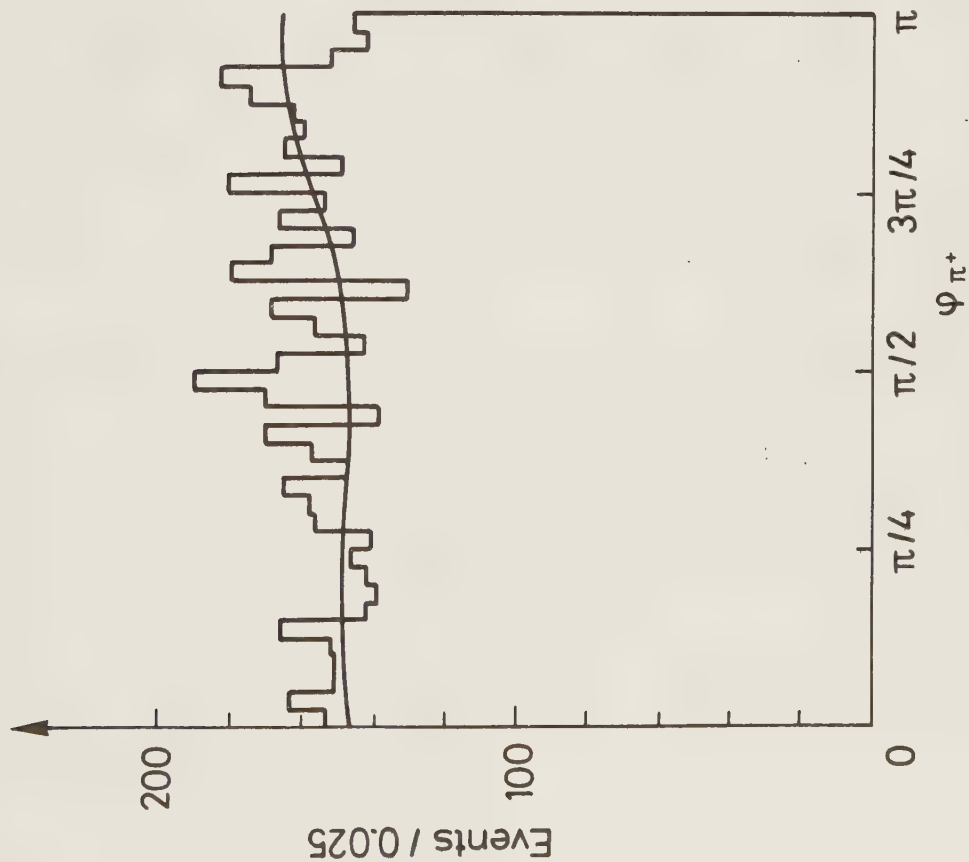


Fig. 6.2

25 GeV/c $\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$
 $m_{3\pi} = 1.0 - 1.2$ GeV
 LIKELIHOOD VERSUS m AND Γ OF $\pi\pi$ RESONANCES

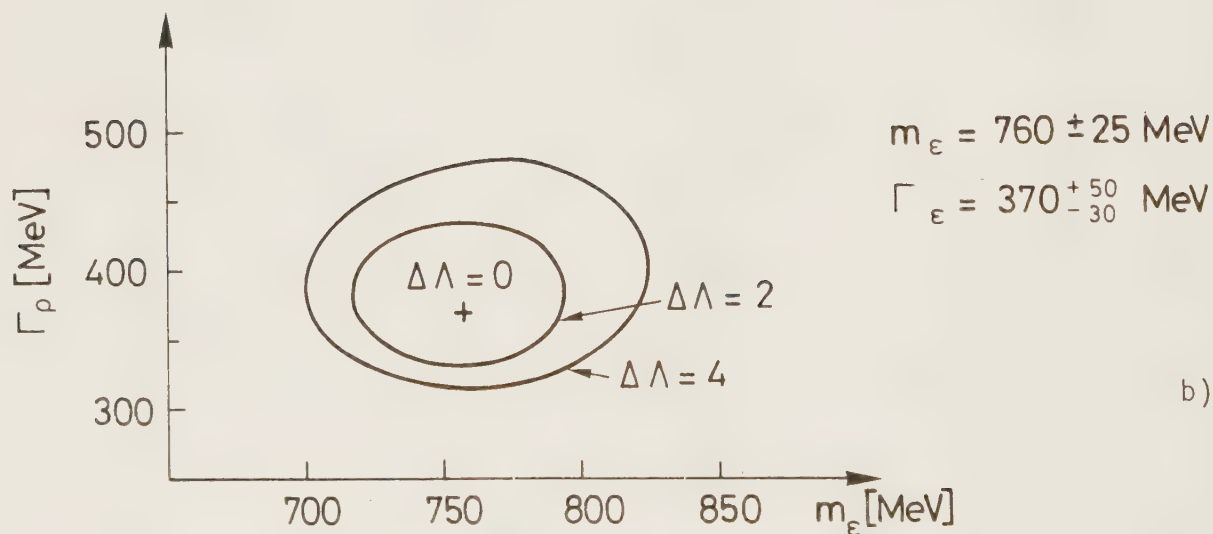
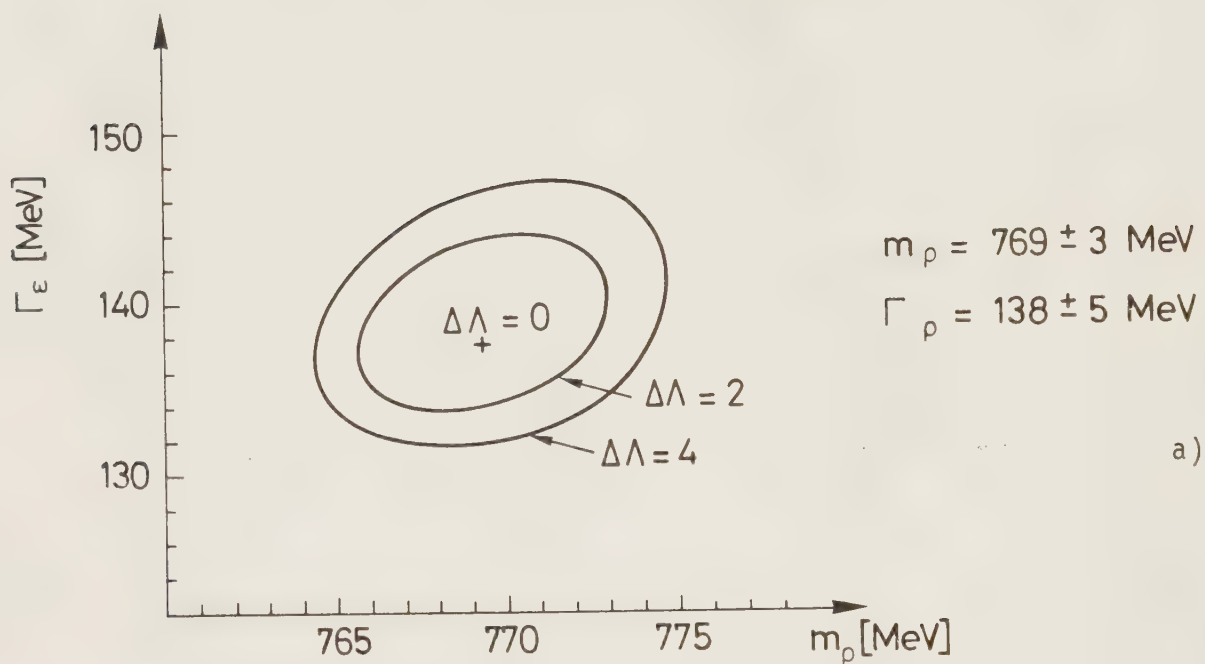


Fig. 6.3

PARAMETRIZATION OF $\pi^+\pi^-$ S-WAVE

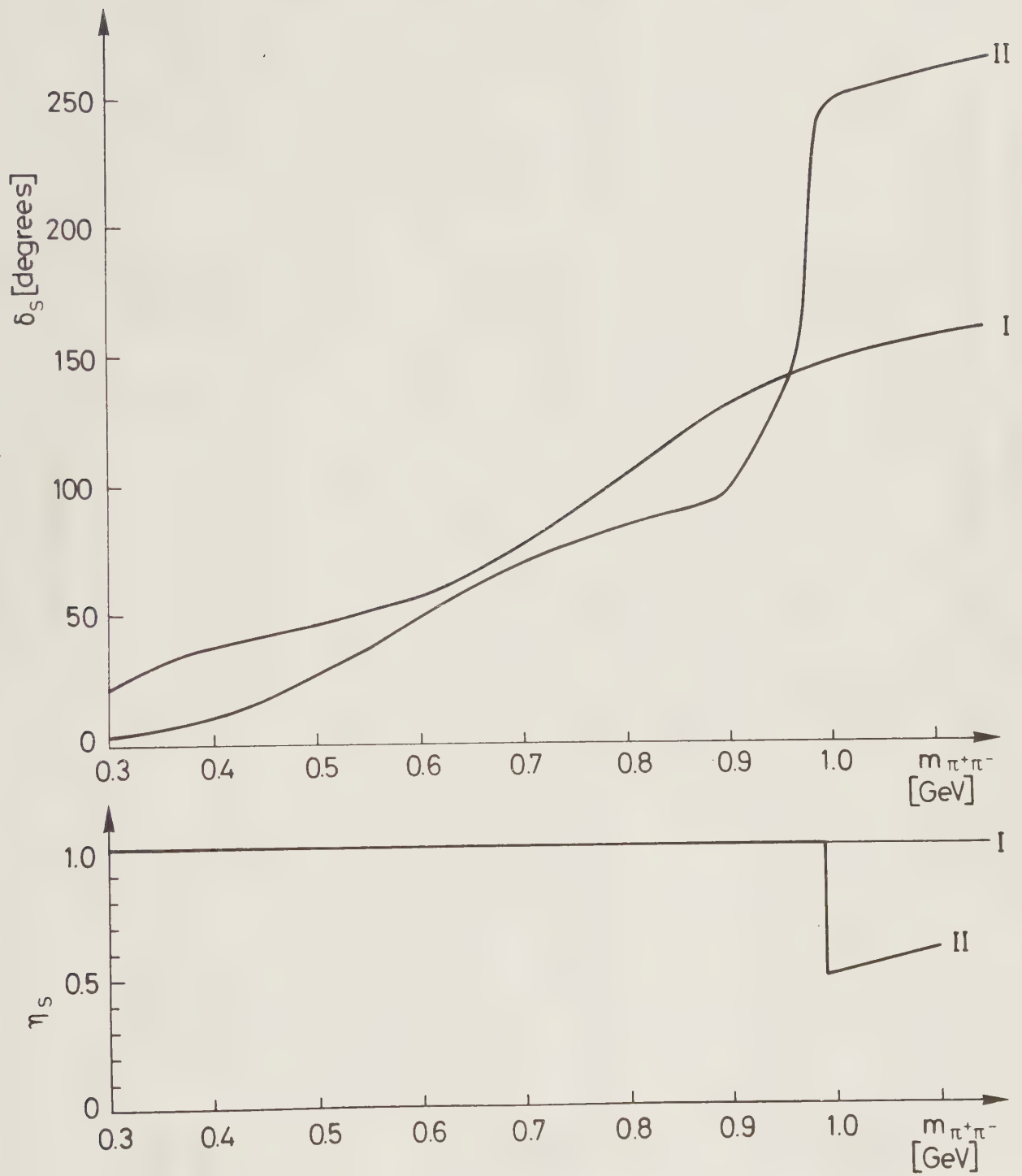


Fig. 6.4

$\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ 25 GeV/c $0.10 < t < 0.30$ (GeV/c)²

PARTIAL WAVE DECOMPOSITION OF THE 3π SYSTEM

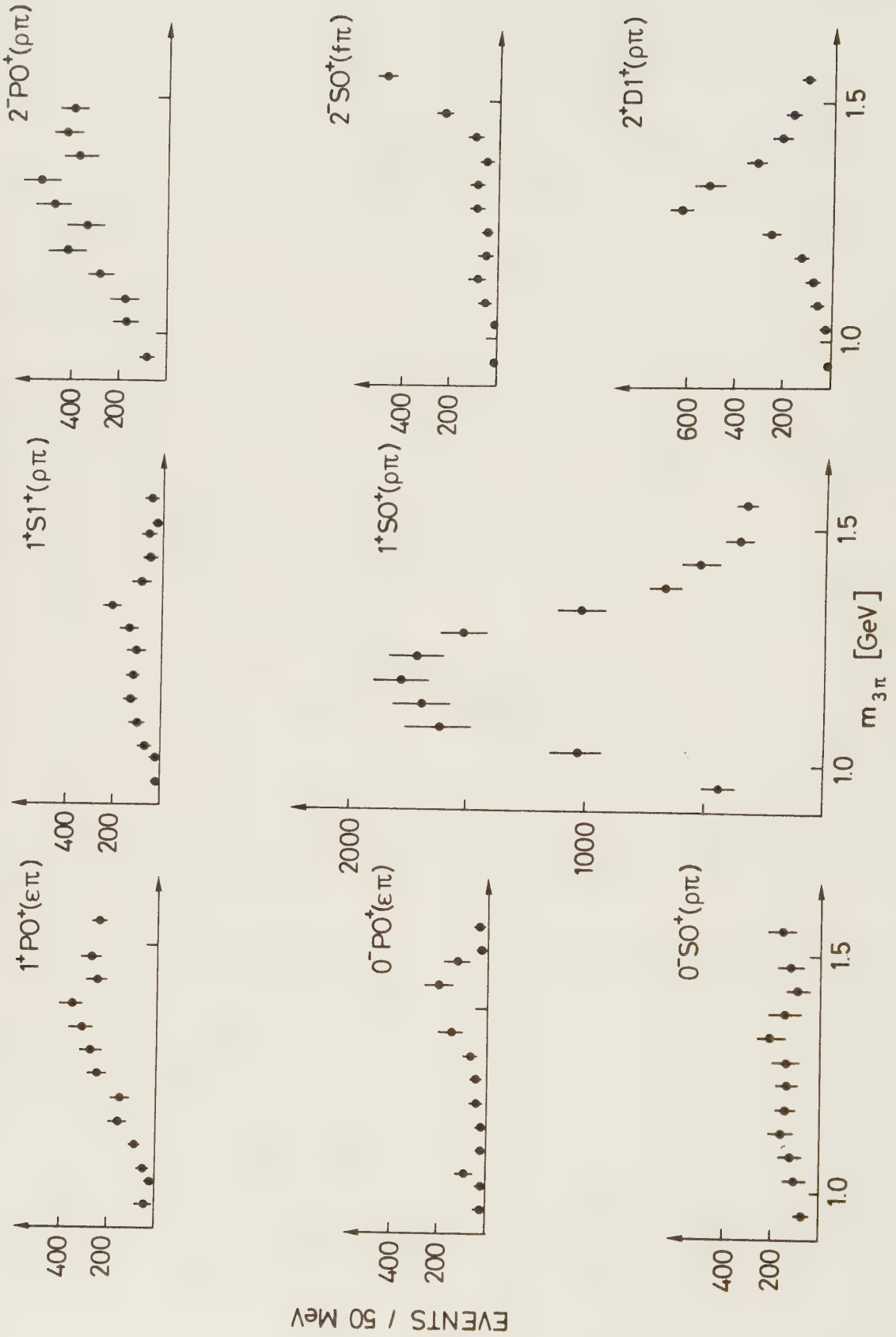


Fig. 6.5 a)

$\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ 4.0 GeV/c $0.04 < t < 0.30$ (GeV/c)²

PARTIAL WAVE DECOMPOSITION OF THE 3π SYSTEM

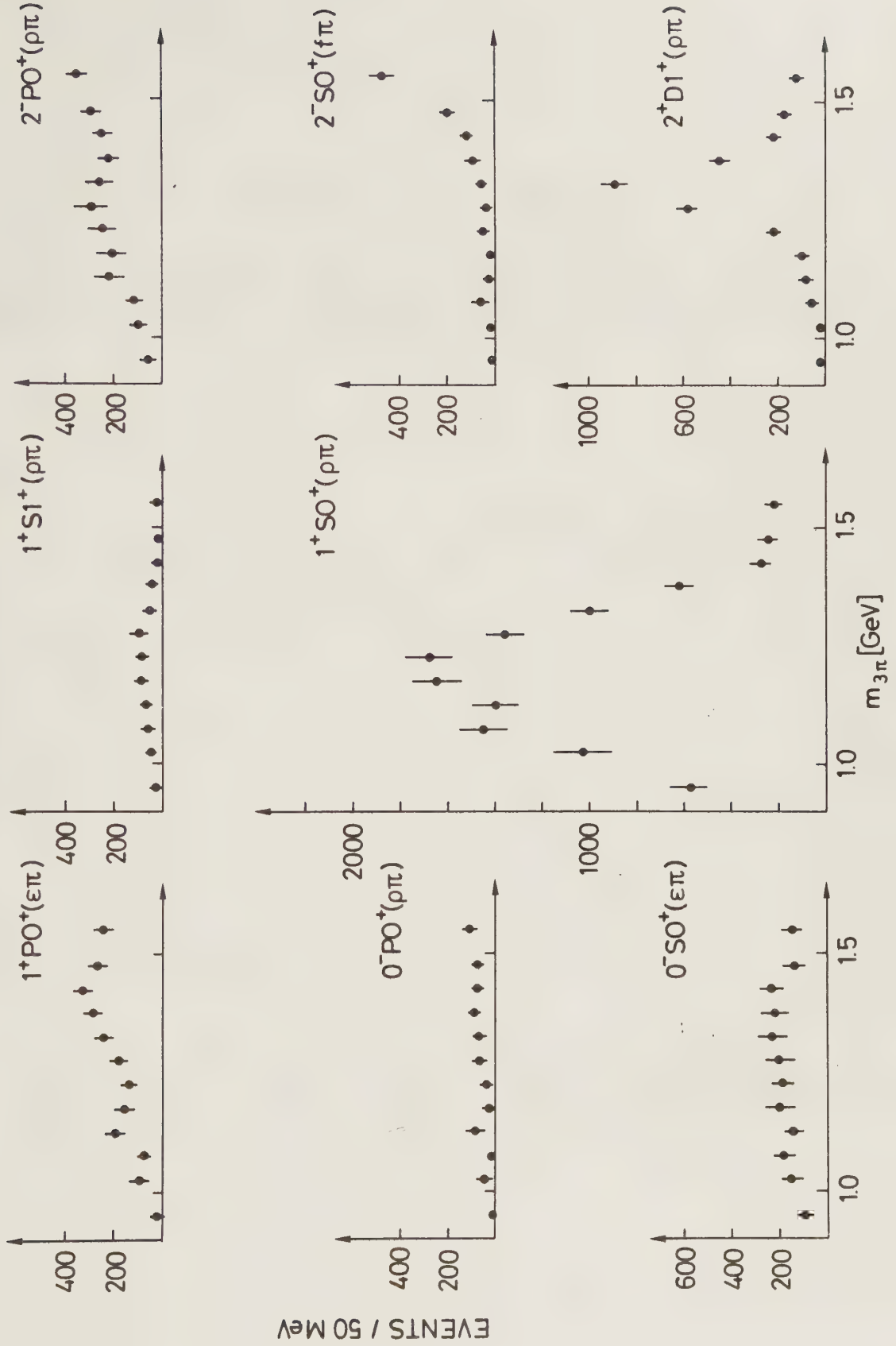


Fig. 6.5 b)

40 GeV/c $\pi^-p \rightarrow \pi^- \pi^- \pi^+ p$

$0.04 < t < 0.33$ GeV/c

VALUES OF 3 BIGGEST EIGENVALUES OF THE 3π DENSITY
MATRIX VERSUS $m_{3\pi}$

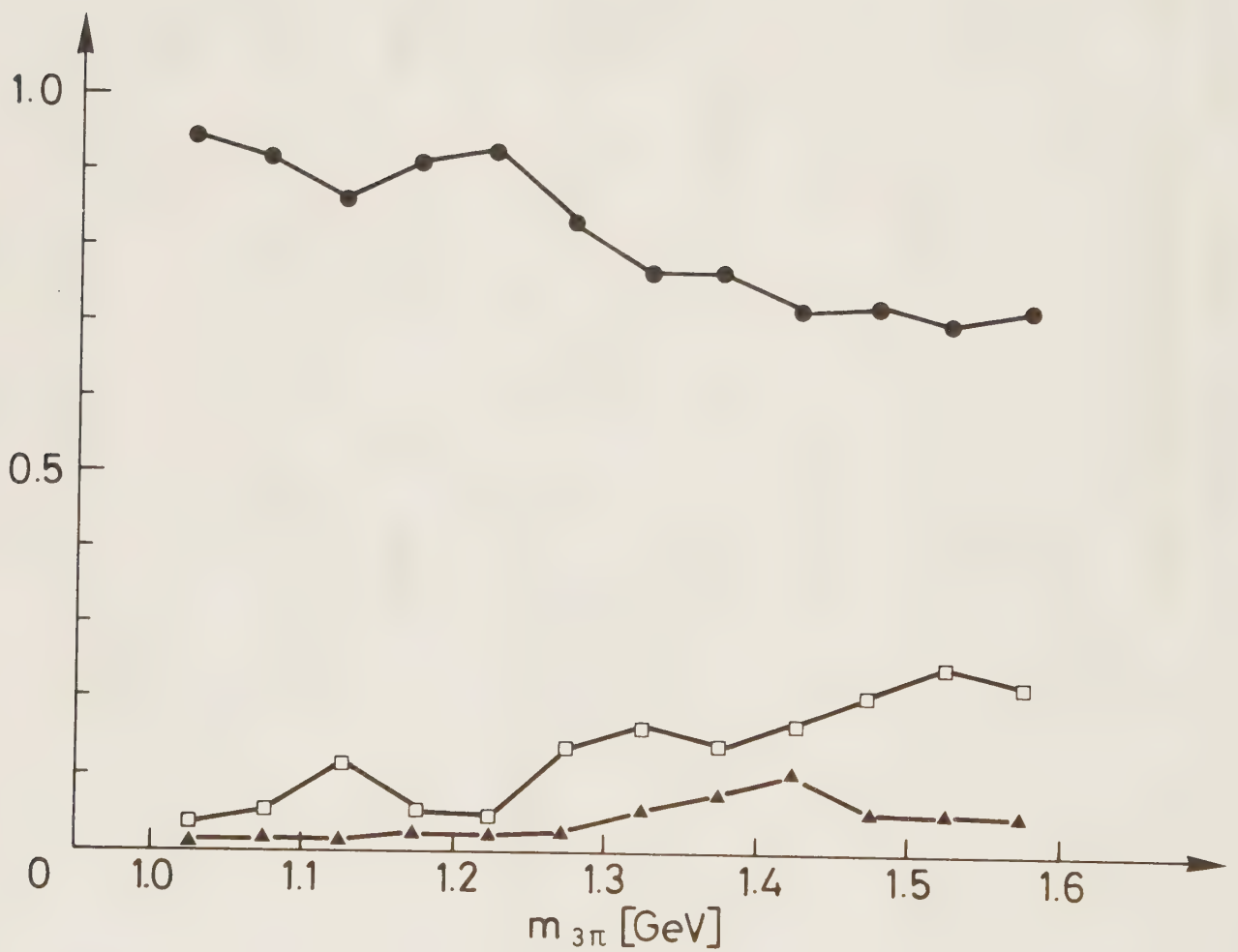


Fig. 6.6

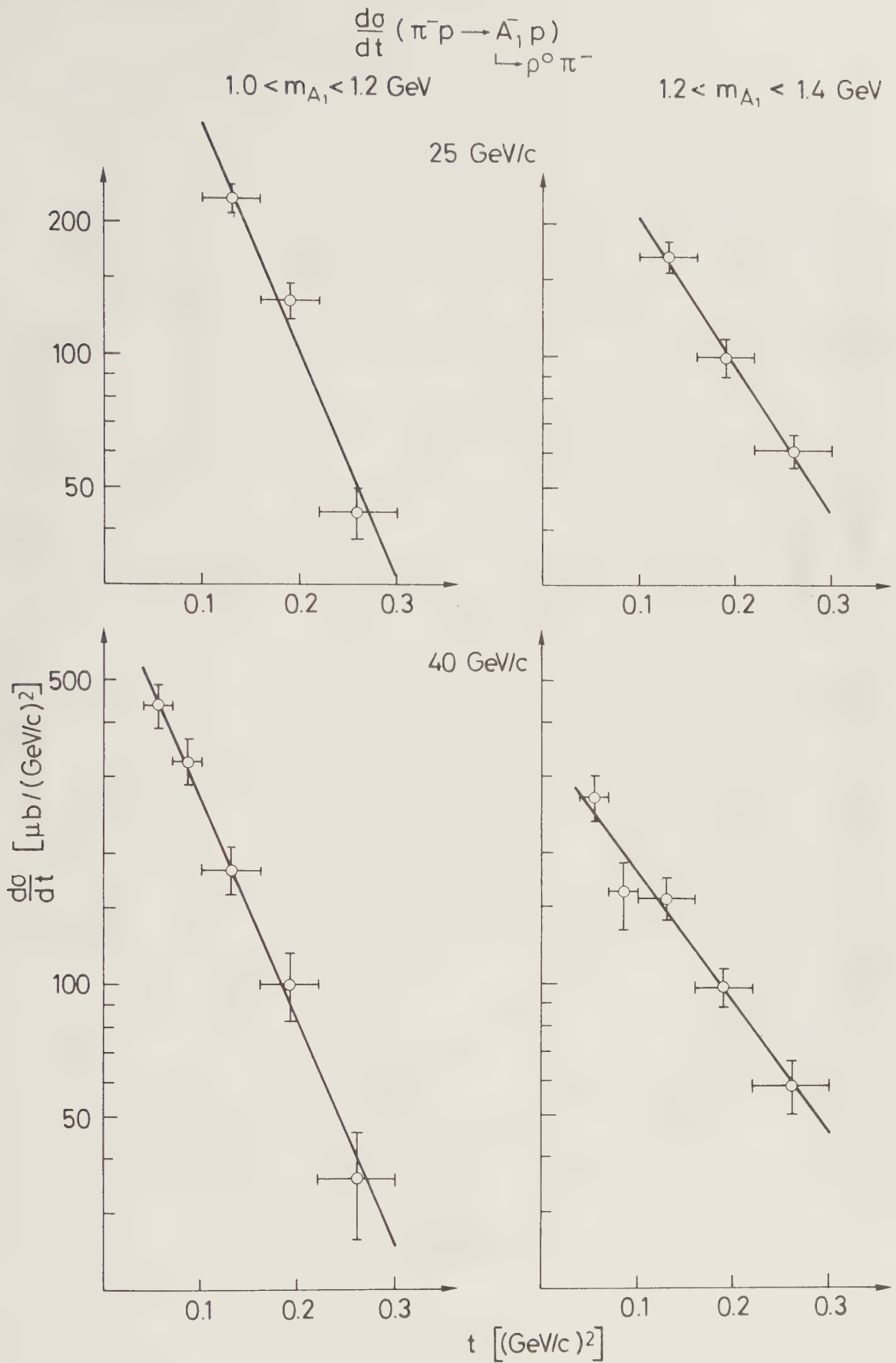


Fig. 6.7

$$40 \text{ GeV } \pi^- p \longrightarrow \pi^- \pi^- \pi^+ p$$

ROTATION ANGLE FROM G.J. SYSTEM TO D-H SYSTEM
FOR 1^+-WAVE

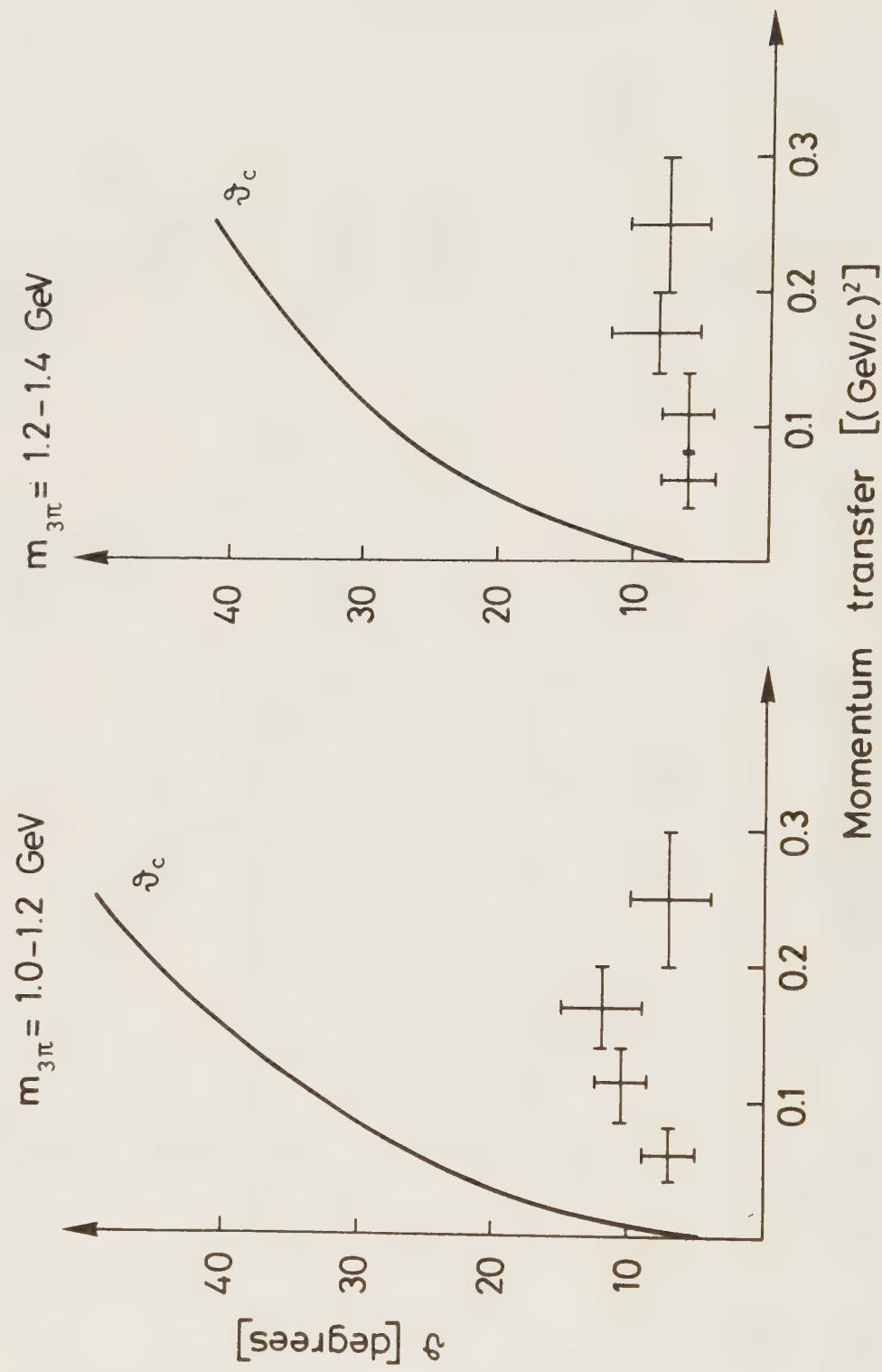
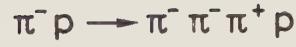


Fig. 6.8



INTERFERENCE ANGLE OF $1^+ S O^+$ WAVE

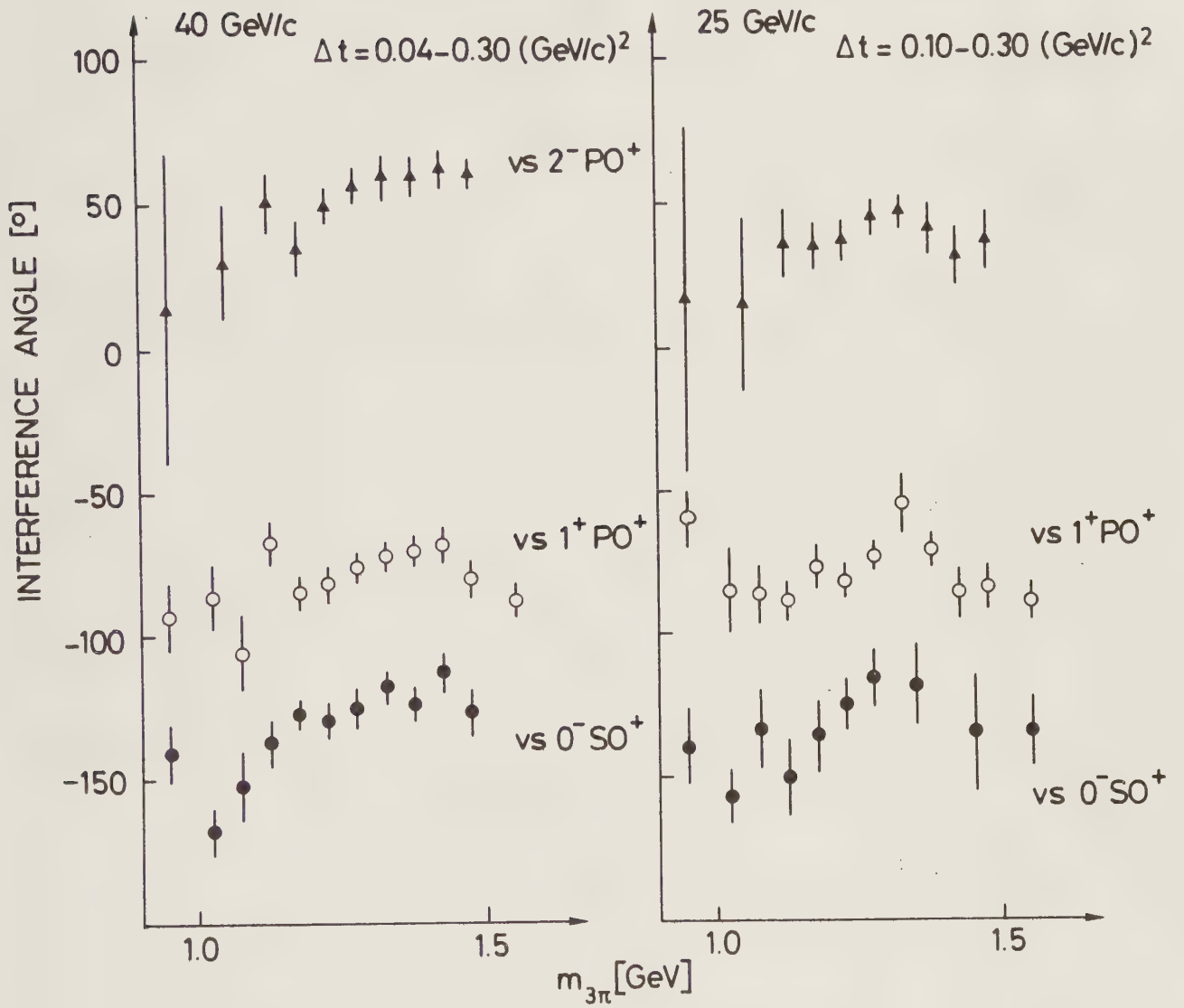


Fig. 6.9

$\pi^- p \rightarrow \pi^- \pi^- \pi^+ p$ 40 GeV/c

$0.17 < t < 0.33 \text{ (GeV/c)}^2$

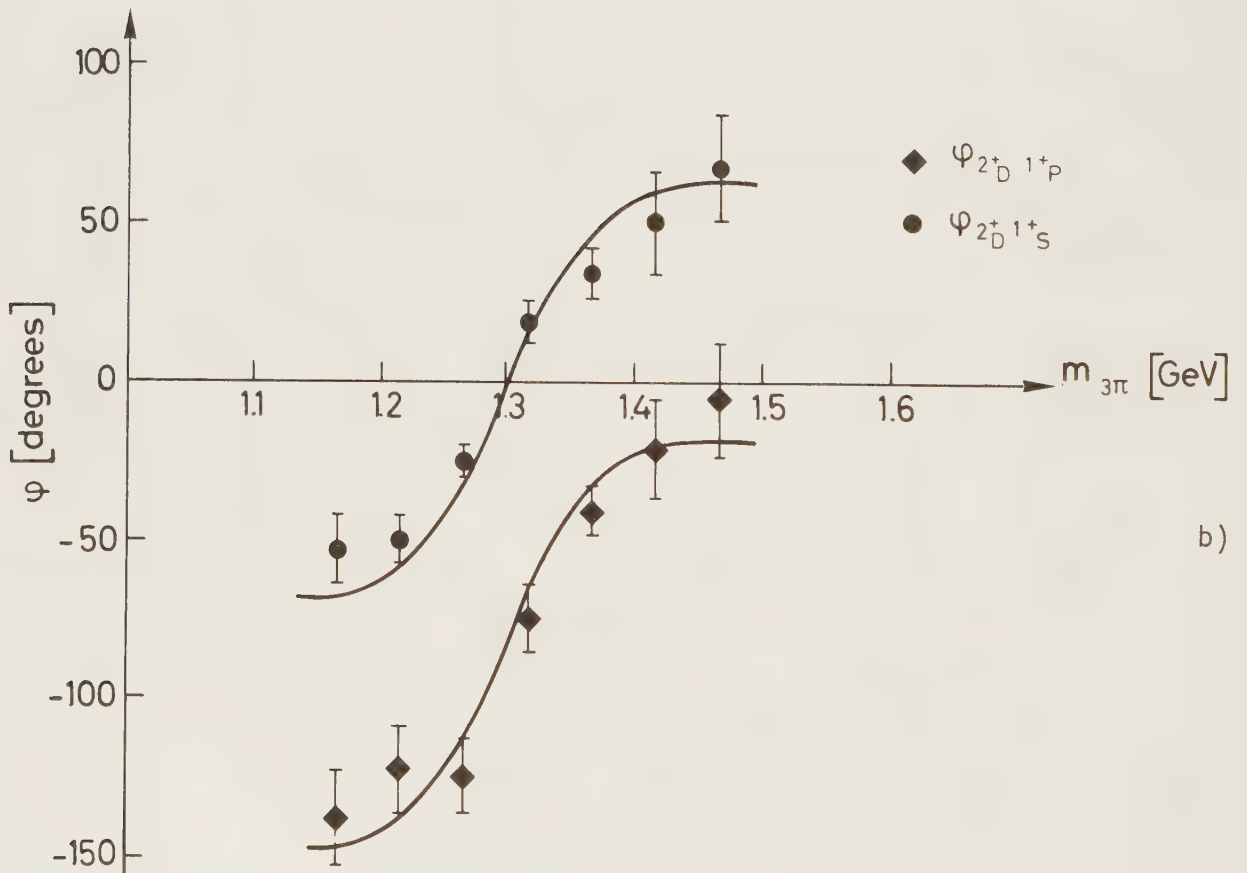
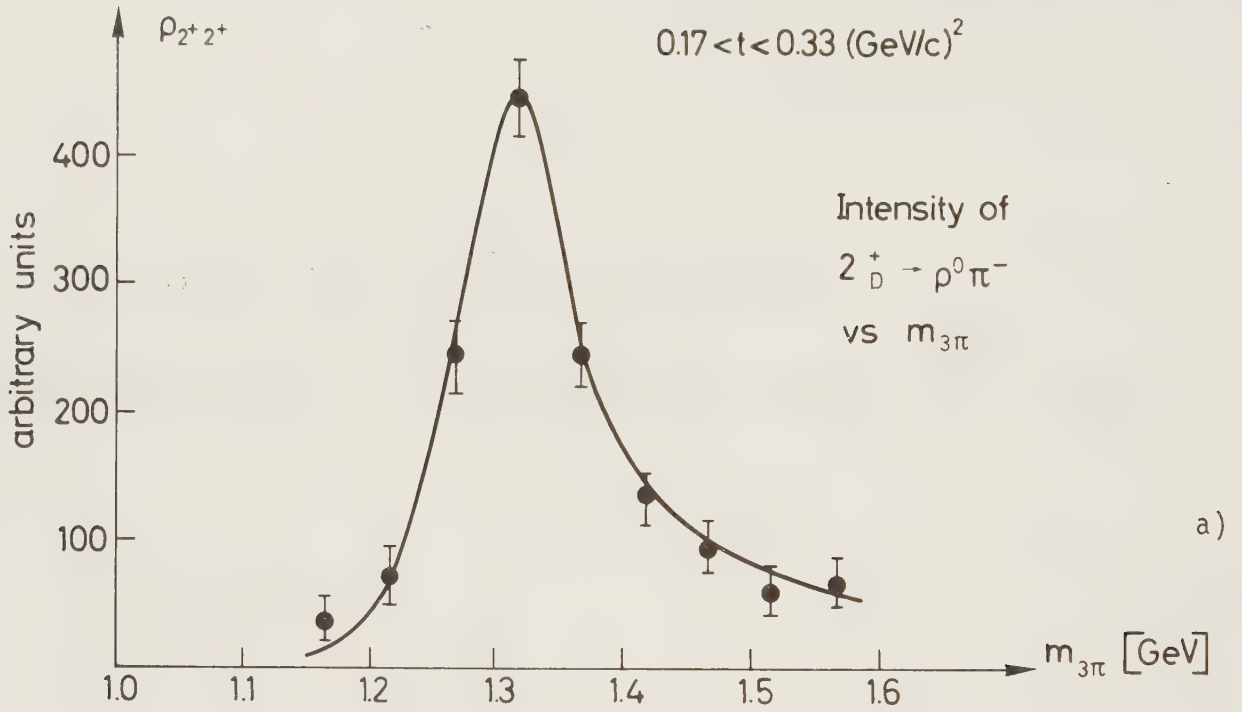


Fig. 6.10

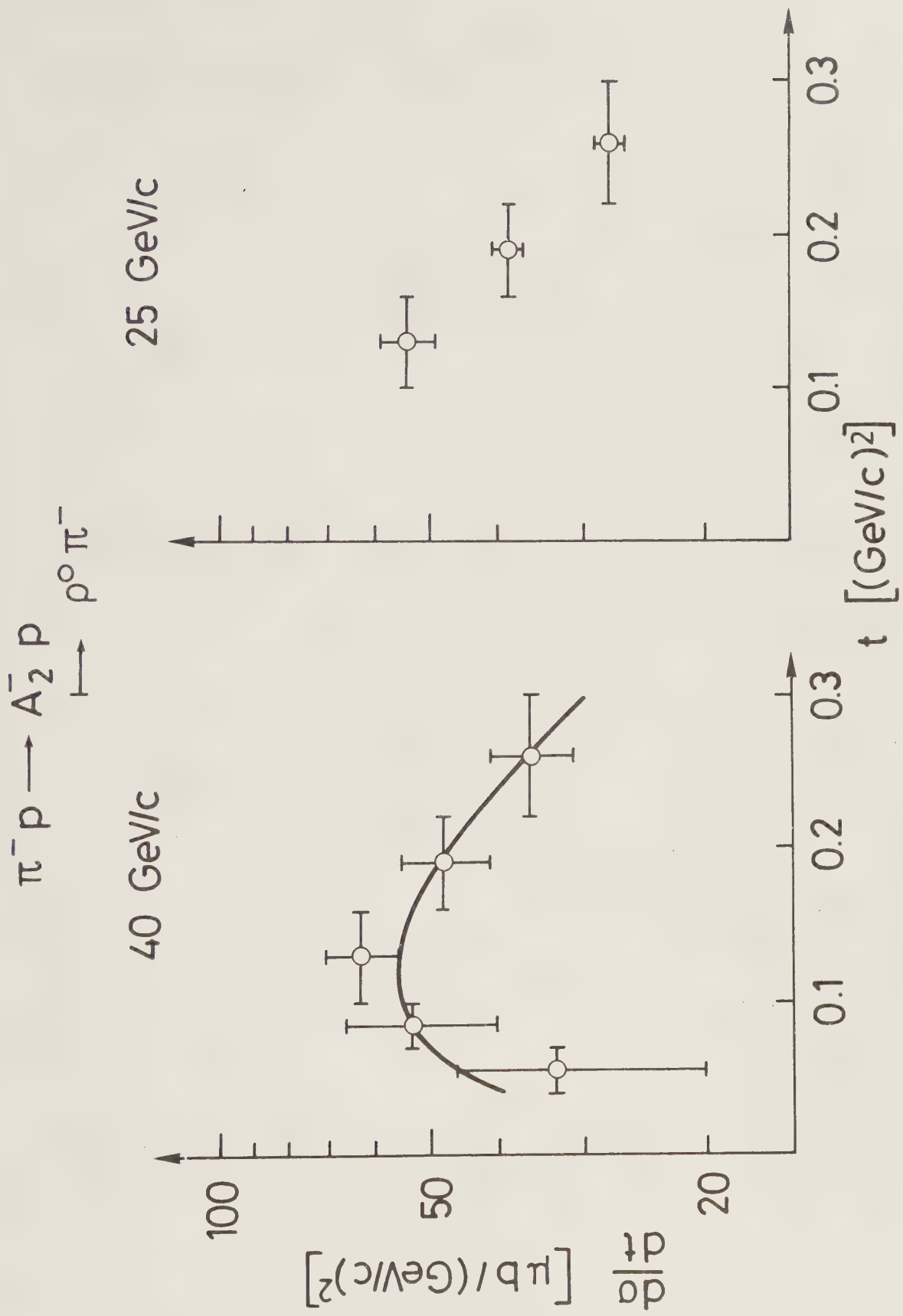


Fig. 6.11

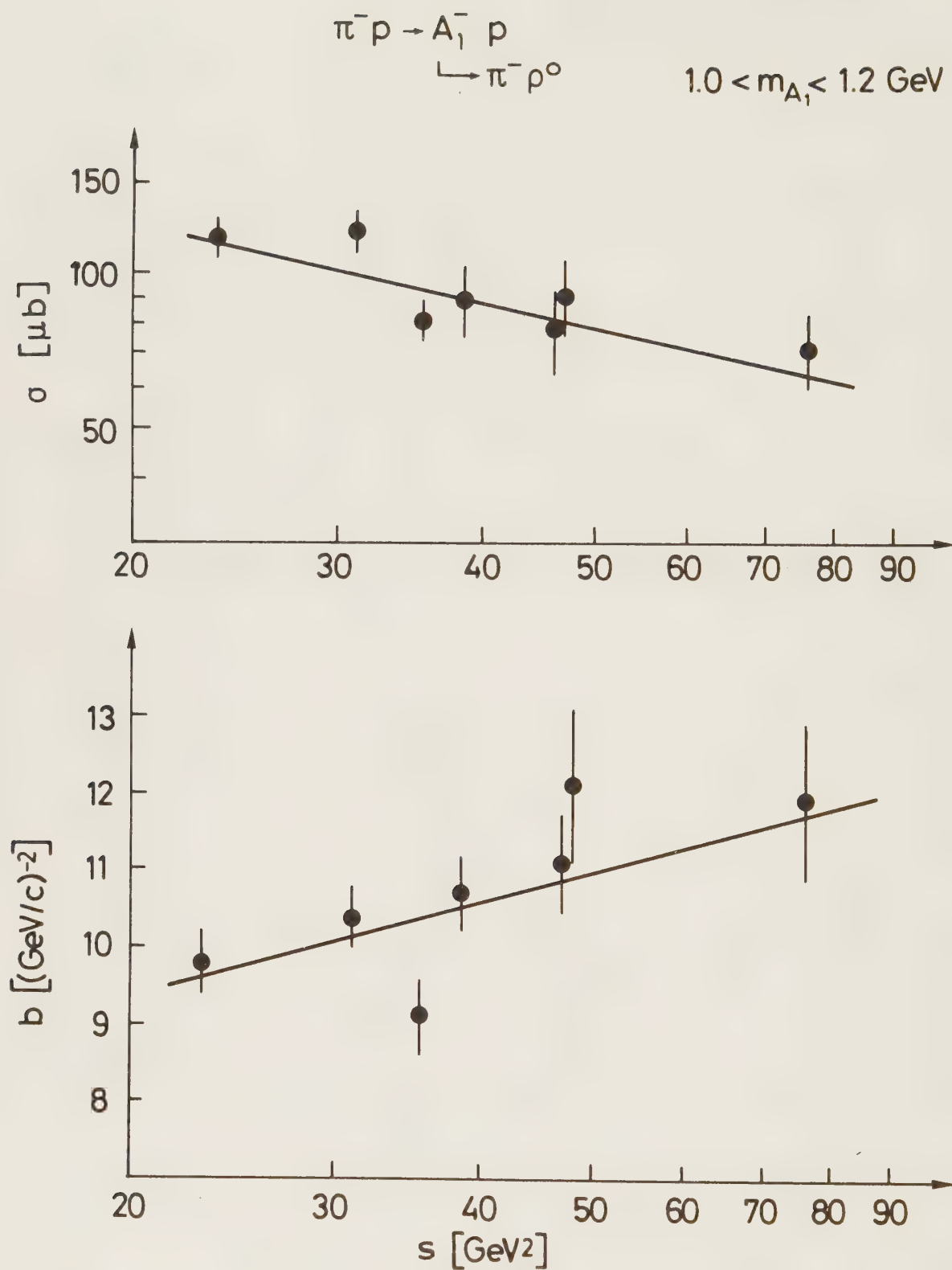


Fig. 6.12

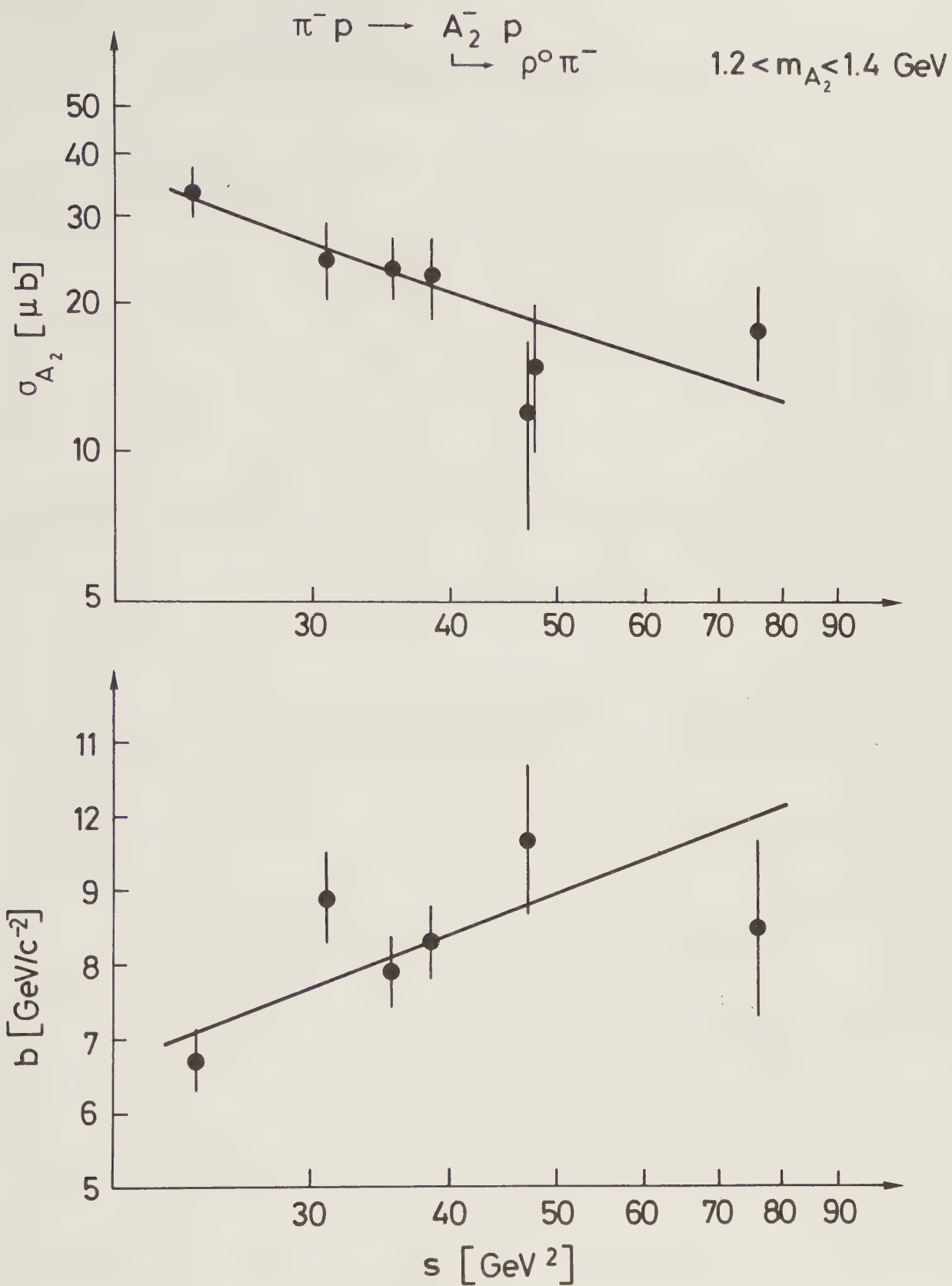
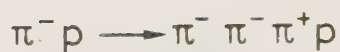


Fig. 6.13



INTERFERENCE OF $1^+(S)$ AND $2^+(D)$ STATES

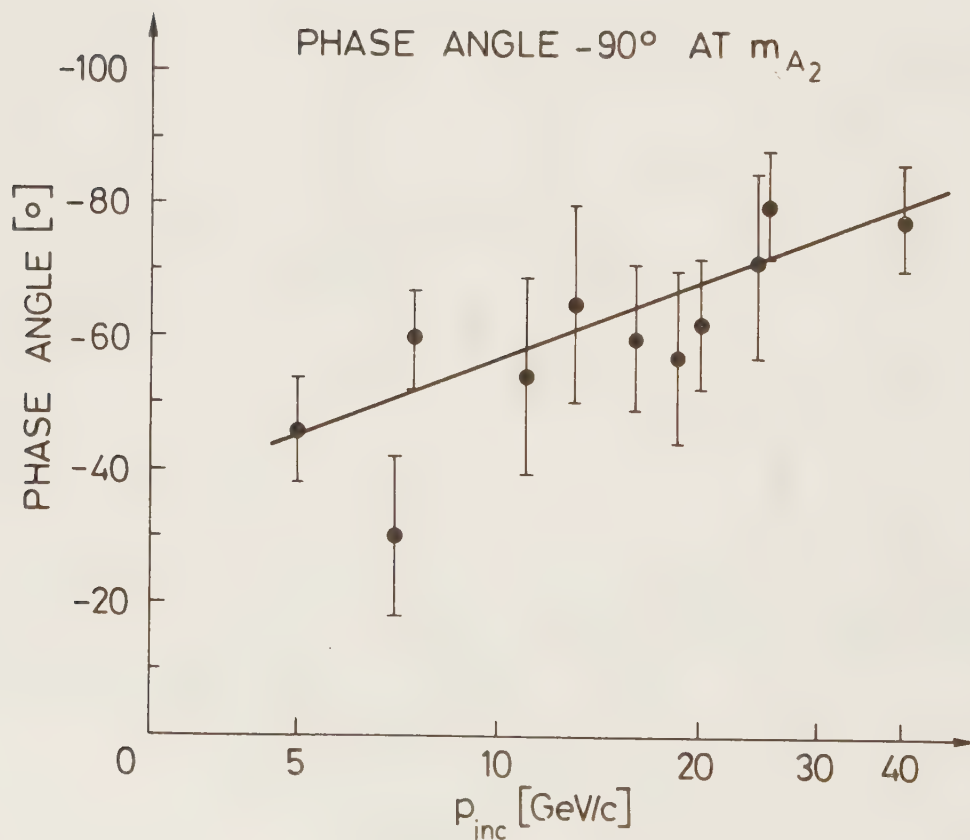
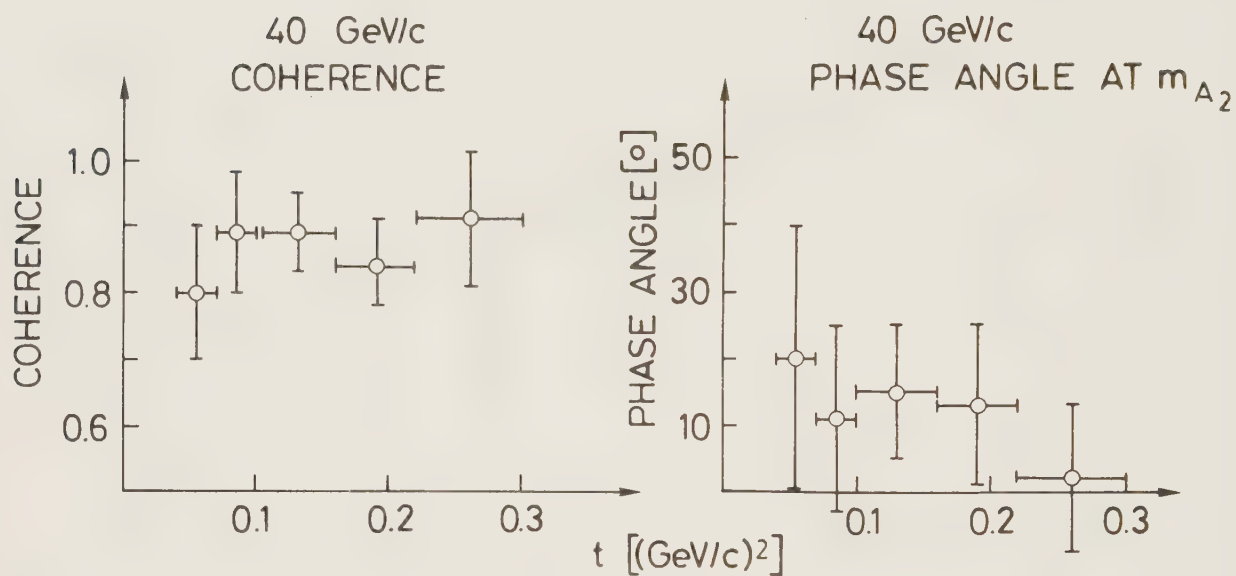


Fig. 6.14

Acknowledgements

Many people have contributed to this work. In particular I should like to express my gratitude to the following:

The members of the Boson Spectrometer Group: Yu.M. Antipov, G. Ascoli, R. Baud, R. Busnello, G. Damgaard, M.N. Kienzle-Focacci, A.A. Lebedev, C. Lechanoine, P. Lecomte, M. Martin, V. Roinishvili, R.D. Sard, A. Weitsch and F.A. Yotch, under the joint leadership of W. Kienzle and L.G. Landsberg. Without their constant help and stimulation this work would not have been possible.

Professor U. Meyer Berkhout, who made my participation in the CERN-IHEP Project possible.

The late Professor P. Preiswerk, and Professors G. Cocconi, B. Gregory and H. Schopper, who initiated and encouraged this 2nd Joint CERN-IHEP experiment.

Doctors Y. Goldschmidt-Clermont and A. Wetherell, as well as Dr. O. Lock and the CERN Administration, for the efficient coordination they carried out.

Professors A.A. Logunov, A.A. Sulayev and Yu.D. Prokoshkin, as well as the Administration of IHEP, for their untiring support and for the warm hospitality offered at Protvino.

Professors E. Picasso and J. Steinberger for the pleasant conditions under which I have been able to work at CERN.

G. Laverrière and E. Leya, as well as our technical staff both at CERN and at IHEP, for their skilful and devoted assistance.

J.D. Hansen, G.T. Jones and M.A. Faessler for many valuable discussions about the Partial Wave Method and their critical reading of the manuscript.

This work was supported in part by the Bundesministerium für Bildung und Wissenschaft of the Federal Republic of Germany.

